Deterministic and Random Growth Processes

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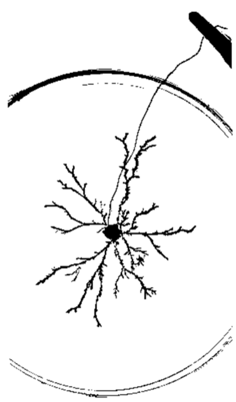
Chapitre 1

An overview

Many natural phenomena are growing processes. Let us mention electrodeposition, lightnings, cracks, colonies of bacterias, percolation, tumors and also towns, networks etc.. The aim of the present monograph is to present the tools



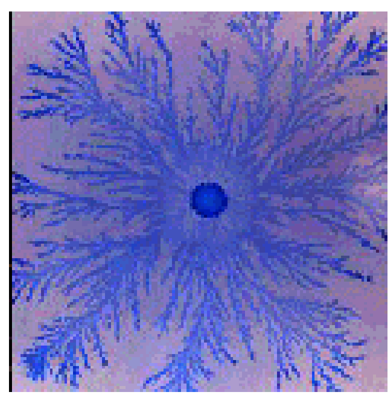


FIG. 1.1 – An experience in electrodeposition. Bacteria colony

to understand 2-dimensional growth processes.
Of course most of the growing processes in nature are 3-dimensional and it would be of great interest to develop a theory valid for any dimension, but it happens that the theory is so developed in dimension 2 and inexistent in dimension ≥ 3 because the main tool is complex analysis, or maybe more precisely conformal analysis, a tool not available in higher dimension.
The processes that we are going to consider will be either deterministic or random. Rather than an abstract introduction of the concepts, let us show how they pop out naturally from examples.

1.1 Hele-Shaw flows

In order to dig oil from an oil mine, the usual method is to inject high pressure water. But water is a low-viscous fluid compared to oil and this leads to highly unstable flow.

This phenomena has been first observed and studied by a British engineer and the turn of the 20th century. His now famous experience consists in filling with glycerine the narrow space between two glass-plates and inject through a small hole high-pressure colored water. The space occupied by water is then a growing cluster showing a dendritic shape with long arms, with a very unstable evolution, with some curved part of the boundary degenerating soon into a long arms while others remain shadowed by the evoluting structure. We will develop the mathe-

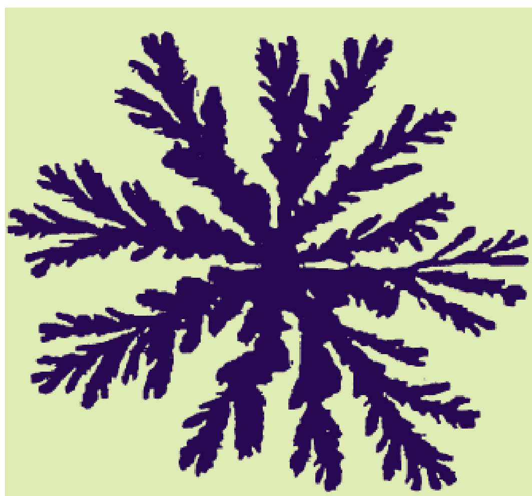


Fig. 1.2 – Hele-Shaw flow

matical theory of this growth later. We briefly mention here that the equation of the motion is a version of Darcy's law : the normal speed of the moving boundary is proportional to the gradient of the pressure function which turns out to be the same as Greens function of the outer domain. As we shall see, this is an ill-posed problem : one way to overcome this difficulty is to regularize the equation. There are physical ways to do this, adding for instance some surface tension : there is also a mathematical way which was introduced by Carleson and Makarov that we will discuss in detail. Another way of handling the ill-posedness is to consider the unstability as some randomness and thus replace the initially deterministic but unstable growth process by a purely random one. The natural random process associated to Hele-Shaw flows is diffusion-limited aggregation (DLA).

1.2 Witten and Sanders DLA-process

Let K_0 be the closed unit disk. We let a disk of radius 1 walk at random from infinity in the plane and we stop it as soon as it touches K_0. The union of the two disks then forms a compact K_1 ; we let them start another disk of radius one to walk at random from infinity and by induction we obtain in this manner a random sequence of compact sets K_n. What can be said about this sequence ? Of course the cluster has a diameter going to ∞. We thus scale it and get a random process which is set-valued with constant diameter (i.e. in such a way that it fits completely inside the computer screen). What can be said about this

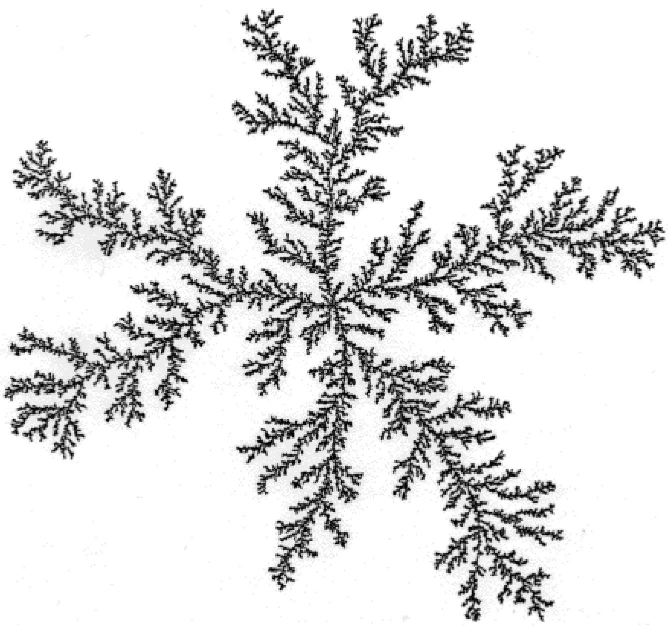


FIG. 1.3 – A DLA cluster

scaled process : does it converge in law ? This is far from being known. Also how does the diameter of the initial process scale as n goes to infinity ? Numerical simulations suggest that $\mathrm{diam}(K_n) \sim n^{1/d}$ for $d = 1.71...$ Kesten has proved that $d \geq 1.5$ (if it exists). In mathematical terms the set K_{n+1} is obtained from K_n by first choosing a point on ∂K_n with the law given by harmonic measure and then attaching a disk at this point. It may be argued that this is not always possible so we modify the model by using conformal mapping. Define for $\theta \in [0, 1[$ and $\delta > 0$ the map $h_{\delta,\theta}$ as the conformal mapping from the complement of the unit disk onto the complement of the unit disk minus the segment $[e^{2i\pi\theta}, (1+\delta)e^{2i\pi\theta}]$ with Laurent expansion $az + ..$, $a > 0$ at ∞ (we say that the conformal mapping is normalized) : the growth process is then given by a starting cluster K_0 and if K_n is defined by a normalized conformal mapping φ_n then K_{n+1} is defined by its

normalized mapping $\varphi_{n+1} = \varphi_n \circ h_{\delta_n,\theta_n}$ with some choice on the constants. For this model to mimic a DLA one must adjust the constants so that the image by φ_n of the segment $[e^{2i\pi\theta}, (1+\delta)e^{2i\pi\theta}]$ has a fixed size : however the model makes sense for any choice of the constants involved. We have thus modelized the growth process by an iteration of simple conformal mappings.
The question is now : how can we get from this discrete model to a continuous one ? The idea is to reduce the size of the attached objects and increase the frequency of attaching time. Concretely this boils down to considering an average over the circle of infinitesimal growth of external rays at this point. To get an idea of what we get in this way let us perform a simple computation. Let h_s be the conformal mapping from the outside of the unit disk onto itself minus the segment $[1, \frac{1+\sqrt{s}}{1-\sqrt{s}}]$. We have $\varphi_{n+1} = \varphi_n \circ h_s$ and we may write

$$\varphi_{n+1}(z) - \varphi_n(z) = z\frac{\varphi_n(h_t)(z) - \varphi_n(z)}{h_s(z) - z}\frac{h_s(z) - z}{z}.$$

If we assume now that t becomes infinitesimal the equation reads

$$\frac{\partial\varphi(z)}{\partial t} = z\frac{\partial\varphi(z)}{\partial z}p(z,t)$$

where

$$p(z,t) = \lim_{s\to 0}\frac{h_s(z) - z}{z} = \frac{z+1}{z-1}.$$

If we had started from a small piece of external ray at $e^{i\theta}$ instead of 1 we would have gotten

$$p(z,t) = \frac{z+e^{i\theta}}{z-e^{i\theta}}$$

instead. To achieve the description of the continuous-time process one finally average this last function over the circle, and the function p becomes

$$p(z,t) = \int_0^{2\pi}\frac{z+e^{i\theta}}{z-e^{i\theta}}d\mu_t(\theta)$$

where μ_t is for each t a probability measure. Notice that this function P is holomorphic outside the disk and has positive real part.
This equation is known as Löwner equation, which is the equation describing the growth of connected clusters. The importance of Löwner equation lies in the fact that it has a converse : one can start with an equation

$$\frac{\partial\varphi(z)}{\partial t} = z\frac{\partial\varphi(z)}{\partial z}p(z,t)$$

where $p(.,t)$ is a one parameter family of holomorphic functions with positive real part and it is true that, under some mild regularity conditions, its solutions actually describe a growth process. This fact is going to be the heart of the matter of these notes. The DLA case corresponds to a function

$$p(z,t) = \frac{z + e^{i\theta(t)}}{z - e^{i\theta(t)}}$$

where θ is a step function. The closure of the set of step functions under uniform limits on compact sets is the set of regulated functions. One of the main contributions of the present monograph will be to characterize geometrically growth processes driven in this sense by regulated functions.

1.3 SLE : Stochastic Löwner Evolution

A major contribution to mathematics has been done by Oded Schramm with his theory of Stochastic Löwner evolution [S]. In its radial version SLE_κ is the Löwner process driven by the function

$$\lambda(t) = e^{i\kappa B_t}$$

where B_t is a standard 1D Brownian motion. This family of random growth processes appeared to be the right tool to describe many scaling limits of discrete processes from statistical mechanics. The most famous one is critical percolation :

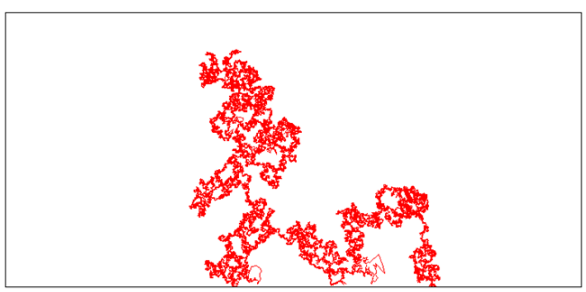


Fig. 1.4 – Chordal SLE_6

consider a half-plane tiled with hexagons. Decide to color black the hexagons of the negative real axis and white those of the positive one. We start to walk along the edges of the hexagon at zero, going up : we meet an hexagon and we toss a coin to decide its colour. If it is black we go right and left if it is right ; we continue in this way, with the difference that it may happen that the hexagon we arrive on is already coloured. But then we simply use the same rule and it will work since it can be easily seen by induction that the last piece of the path is always between two different colours. Moreover a moment's reflection shows that the path we obtain is simple and goes to infinity. The question we adress is :

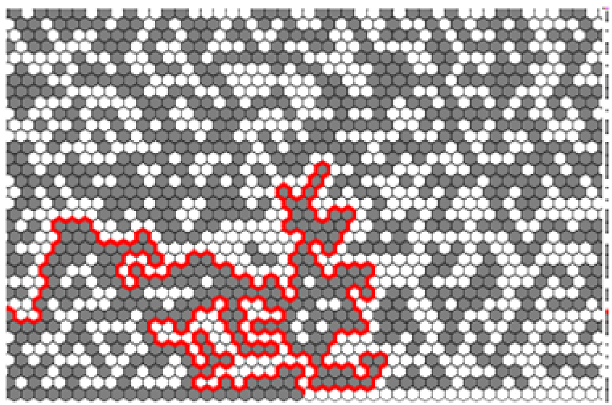


Fig. 1.5 – Critical Percolation

what happens when the mesh of the lattice converges to 0 ? The problem here is two fold (not speaking about the delicate problem of what we exactly mean by convergence). Lawler,Shramm and Werner have proved [] that if we let the mesh go to 0 then it must converge to a SLE process if this limit has a property called conformal invariance. It was Smirnov [] who proved this propertyTh fact that $\kappa = 6$ follows from the locality property of this process (see below). It has also been shown [] that SLE_2 is the scaling limit of LERW (loop-erased random walks) and many other similar results of this kind were either proved or conjectured [][]. But perhaps the more spectacular achievment of this theory has been the proof of Mandelbrot conjecture by Lawler, Schramm and Werner : if U stands for the unbounded component of $\mathbb{C}\backslash B([0,1])$ where B_t is a planar Brownian Motion then almost surely ∂U has dimension 4/3. One of the tasks of these notes will be to provide a complete proof of this fact.

1.4 Growth of cities

It has been soon realized that cities do not grow like a DLA. This is due to the fact that density of population as a function from the center would decay like the inverse of a polynomial while it is known that for most of the cities it does decrease exponentially, as

$$\rho = \rho_o e^{-\lambda r},$$

r being the distance to the center. So we replace DLA by percolation theory : we

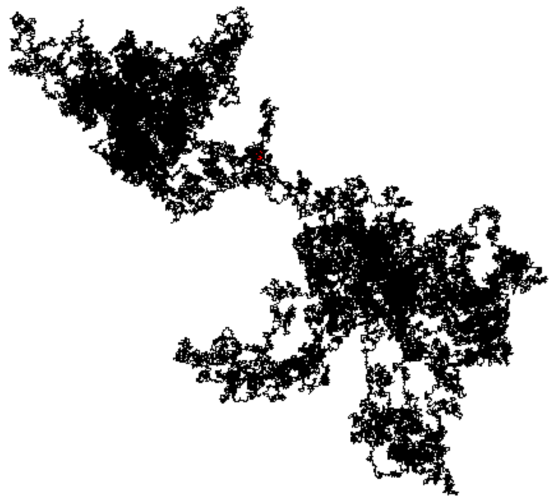


FIG. 1.6 – The Brownian Frontier

FIG. 1.7 – London

consider more specifically a Gaussian distribution on $\mathbb{Z}^2$ such that

$$cov(X_z, X_\zeta) \sim |z-\zeta|^{-\alpha/2},\ |z-\zeta| \geq 1,$$

and such that the X_z's are identically distributed. Define $p(r) = e^{-\lambda r}$ and let $\theta(r)$ be the unique real number such that $p(r) = P(X_z \leq \theta(r))$. Then for each $\omega \in \Omega$

we color points of the lattice black if $X_z \leq \theta(r)$. This defines a "random city" with concentration ρ. Letting now $\lambda \to 0$ we get an increasing family of clusters, modelling the growth by the postulate that concentration grows with the time, allowing to replace time by concentration.
We will also discuss a similar construction by Zabrodin etal : this models a cloud of electrons in a magnetic field : if we decrease the strength of this magnetic field the clouds increases and this growth is a Hele-Shaw flow.

Chapitre 2

Background in Complex Analysis

2.1 Simply connected Riemann surfaces

An arc in a metric space X is a continuous mapping γ from some interval $[a,b] \subset \mathbb{R}$ in X. Such an arc is said to be closed if $\gamma(a) = \gamma(b)$. Two arcs γ_1, γ_2 defined on the same interval $[a,b]$ are said to be homotopic if there exists $\Gamma : [a,b] \times [0,1] \to X$ continuous such that

$$\forall s \in [a,b],\, \Gamma(s,0) = \gamma_1(s)\,,\, \forall s \in [a,b],\, \Gamma(s,1) = \gamma_2(s).$$

Definition 2.1.1. *: The space X is called simply connected if it is connected and if every closed arc $\gamma : [a,b] \to X$ is homotopic to the constant arc $\gamma_0 : t \in [a,b] \mapsto \gamma(a)$.*

When X is a plane domain we have the following equivalent characterizations of simply connected domains :

Theorem 2.1.1. *: For a connected open subset Ω of $\mathbb{C}$ the following are equivalent :*
(1) Ω is simply connected,
(2) $\overline{\mathbb{C}} \backslash \Omega$ is connected,
(3) For any closed arc γ whose image lies in Ω and any $z \notin \Omega$, $\mathrm{Ind}(z,\gamma)$=0.

We recall that $\mathrm{Ind}(z,\gamma)$ stands for the variation of the argument (mesured in number of turns) of $\gamma(t) - z$ along $[a,b]$. When γ is piecewise C^1 this quantity is also equal to

$$\frac{1}{2i\pi}\int_a^b \frac{\gamma'(s)\mathrm{ds}}{\gamma(s)-z} = \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Let us sketch the proof of this theorem. Assume (1) holds but not (2) : then there exists a connected component F of $\overline{\mathbb{C}} \backslash \Omega$ not containing ∞ and one can build a simple arc Γ surrounding F. This arc cannot be homotopic to a point since

$$\forall z \in F,\, Ind(z,\Gamma) = 1.$$

If (2) holds but not (3) then there exists an arc γ in Ω and a point $z \notin \Omega$ with $Ind(z, \gamma) \neq 0$. Such a point cannot be in the unbounded component of the complement of the image of γ and $\overline{\mathbb{C}}\backslash\Omega$ cannot be connected.
The fact that (3) implies (1) will be a consequence of the Riemmann mapping theorem below. Before we come to this point let us point out two properties of domains satisfying (3).
Let f be a holomorphic function defined on a subdomain Ω of $\mathbb{C}$ satisfying (3). By global Cauchy theorem and (3)

$$\int_{\gamma} f(z)\mathrm{d}z = 0. \tag{2.1}$$

for every closed arc γ of Ω. Fix now $z_0 \in \Omega$. Since Ω is arcwise connected, for every $z \in \Omega$ there exists an arc $\gamma : [a, b] \to \Omega$ such that $\gamma(a) = z_0, \gamma(b) = z$. We define $F(z) = \int_{\gamma} f(z)\mathrm{d}z$. By (3.3), this definition is independent of the choice of the arc γ as soon as it joins z_0 to z inside Ω. It is easy to check that $F' = f$. We have thus proven that every holomorphic function in a simply connected domain admits a holomorphic anti-derivative. This has a converse :

Theorem 2.1.2. *: A domain Ω in $\mathbb{C}$ satisfies(3) if and only if every holomorphic function in Ω has an anti-derivative.*

Proof : : We have just proven the $\Rightarrow$ part. To prove the converse let $z_0 \in \mathbb{C}\backslash\Omega$. Then $\frac{1}{z-z_0}$ is holomorphic in Ω and so has an anti-derivative. Cauchy's theorem applied to this function implies that $\mathrm{Ind}(z_0, \gamma) = 0$ for every closed arc in Ω.

Consider in particular a non-vanishing holomorphic function f and let w_0 be any complex number such that $e^{w_0} = f(z_0)$. Define g on Ω as the unique antiderivative of $\frac{f'}{f}$ such that $g(z_0) = w_0$. Then $(fe^{-g})' = (f' - g'f)e^{-g} = 0$ and thus $e^g = f$. We have thus proved that every non-vanishing holomorphic function in a simply-connected domain admits a holomorphic logarithm and thus also a holomorphic determination of its square root.

A Riemann surface is a one-dimensional complex manifold. The plane or any open subset of the plane are obviously Riemann surfaces. The first non-planar Riemann surface is the Riemann sphere : as a topological space it is the one-point compactification of $\mathbb{C}$, and we will denote it by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. To define a complex manifold structure on it, we only need to define local coordinate at infinity, but this is just $z \mapsto 1/z$ defined outside 0. For further purposes let us notice that the theorem 2.1.1 remains true for the Riemann sphere.
The uniformization theorem of Klein-Poincaré asserts that every simply connected Riemann surface is isomorphic (as a complex manifold) to the plane, the unit disk or, if it is compact, to the Riemann sphere. In the next paragraph we will prove a special case of this theorem, that is that every proper simply-connected subdomain of the plane is isomorphic to the unit disk.

2.2 Riemann Mapping Theorem

Theorem 2.2.1. *(Riemann) : Let Ω be a simply-connected proper subdomain of $\mathbb{C}$ and $w \in \mathbb{C}$. Then there exists a unique biholomorphic map $g : \Omega \to \mathbb{D}$ such that $g(w) = 0, g'(w) > 0$. (Here and in the future $\mathbb{D}$ will stand for the unit disk).*

An equivalent statement is that there exists a unique holomorphic bijection $f : \mathbb{D} \to \Omega$ sending 0 to z_0 and such that $f'(0) > 0$. This specific map f will be called the Riemann map for z_o.

Proof :

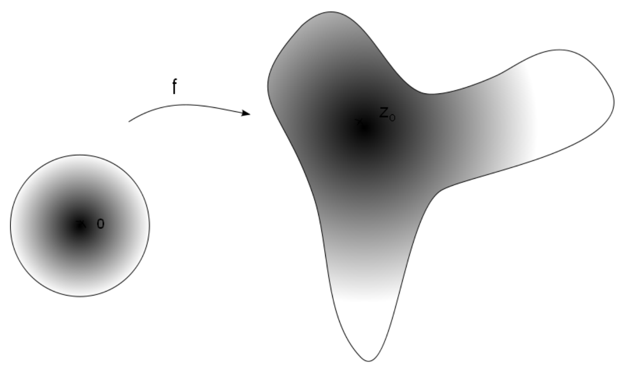


Fig. 2.1 – Riemann map

1) Uniqueness : Suppose g_1, g_2 do the job ; then $\phi = g_2 \circ g_1^{-1} : \mathbb{D} \to \mathbb{D}, 0 \mapsto 0, \phi'(0) > 0$ and the same is true for ϕ^{-1} : Schwarz lemma then implies that $\phi(z) = z, z \in \mathbb{D}$.
2) Existence : Let $E = \{g : \Omega \to \mathbb{D}$ holomorphic and injective with $g(w) = 0, g'(w) > 0\}$. Let us first prove that E is not empty. To do so, we consider $z_0 \in \mathbb{C}\backslash\Omega$. Then $z \mapsto \frac{1}{z-z_0}$ is a non vanishing holomorphic function in Ω and thus admits a square root we denote h. Since h is open there exists $\varepsilon > 0$ such that $B(h(w), \varepsilon) \subset h(\Omega)$. Then we must have $B(-h(w), \varepsilon) \cap h(\Omega) = \emptyset$ because if not there would exist $\zeta \in \Omega$ such that $h(\zeta) \in B(-h(w), \varepsilon)$. Let $\Omega_1 = h^{-1}(B(h(w), \varepsilon))$; ζ cannot belong to Ω_1 because $B(h(w), \varepsilon) \cap B(-h(w), \varepsilon) = \emptyset$ but on the other hand there exists $\zeta' \in \Omega_1$ such that $h(\zeta') = -h(\zeta) \Rightarrow h(\zeta)^2 = h(\zeta')^2$ thus contradicting the injectivity of h^2. We can then define

$$g(z) = \frac{\varepsilon}{h(z) + h(w)}$$

which sends Ω into $\mathbb{D}$: precomposing with a judicious Möbius transformation, we get an element of E. If $g \in E$ we can consider

$$g^*(z) = g(w + zd(w, \partial\Omega))$$

mapping the unit disk into itself and 0 to 0. Applying Schwarz lemma to g^* we see that

$$g'(w) \leq \frac{1}{d(w, \partial\Omega)}$$

if $g \in E$. Let then $M = \sup\{g'(w), g \in E\}$ and (g_n) a sequence of elements of E such that $g_n'(w) \to M$. It is a normal family so, taking if necessary a subsequence we may assume that (g_n) converges uniformly on compact sets to some map g with $g(w) = 0, g(\Omega) \subset \overline{\mathbb{D}}$. Moreover $g'(w) = M$ so that in particular g is not constant. By Hurwitz theorem it must be injective and open and we have proven that $g \in E$.
To finish the proof it suffices to show that g is onto $\mathbb{D}$.
Suppose not : let $z_0 \in \mathbb{D} \backslash g(\Omega)$. Let

$$h(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

be an automorphism of the disk such that $h(z_0) = 0$. The mapping $h \circ g$ is one to one and non vanishing ; there thus exists γ holomorphic and injective in Ω such that $\gamma^2 = h \circ g$. If k is the automorphism of the disk such that $k(\gamma(w)) = 0, k'(\gamma(w))\gamma'(w) > 0$ then $\tilde{g} = k \circ \gamma \in E$. We want to compute $\tilde{g}'(w)$. We compute $|h'(0)| = 1 - |z_0|^2$ and $|h(0)| = |z_0|$. Also

$$|k'(\gamma(w))| = \frac{1}{1 - |\gamma(w)|^2} = \frac{1}{1 - |z_0|}.$$

We have $2\gamma(w)\gamma'(w) = h'(0)g'(w)$ and hence

$$\tilde{g}'(w) = |k'(\gamma(w))||\gamma'(w)| = \frac{(1 + |z_0|)g'(w)}{2\sqrt{|z_0|}}.$$

But $1 + |z_0| > 2\sqrt{|z_0|} \Rightarrow \tilde{g}'(w) > g'(w) = M$, contradicting the maximality of M.

2.2.1 Domains containing ∞.

We will often consider domains of the form $\Omega = \overline{\mathbb{C}} \backslash K$ where K is compact, connected, with connected complement (we say then full), containing 0 but not reduced to this point. We will call to simplify such a set a CCF-set and its complement a CCF-domain. These domains are by (2.1.1) precisely the simply-connected subdomains of $\overline{\mathbb{C}}$ containing ∞ but not 0, and furthermore $\neq \overline{\mathbb{C}} \backslash \{\infty\}$. In order to state a Riemann mapping theorem for these domains we consider the reference $CCF-$domain $\Delta = \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$: we also recall holomorphicity at ∞ for a mapping fixing ∞, using the complex structure at ∞.

Definition 2.2.1. *If $\Omega = \overline{\mathbb{C}} \backslash K$ where K is a CCF-compact, and $f : \Omega \to \overline{\mathbb{C}} \backslash \{0\}$ is a mapping fixing ∞, we say that f is holomorphic at ∞ if the mapping*

$$\tilde{f}(z) = \frac{1}{f(\frac{1}{z})}$$

is holomorphic at 0.

If f is holomorphic at ∞ then

$$\frac{f(z)}{z} = \frac{\frac{1}{z}}{\tilde{f}(\frac{1}{z})}$$

has a limit as z tends to ∞ which is equal to $1/\tilde{f}'(0)$ and denoted $f'(\infty)$. In particular $f'(\infty) \neq 0$ if f is injective and f has the asymptotic development (at ∞)

$$f(z) = f'(\infty)z + c_0 + \frac{c_1}{z} + ...$$

We may now state the adapted version of Riemann mapping theorem :

Theorem 2.2.2. *If K is a CCF-compact there exists a unique holomorphic bijection*

$$f : \Delta \to \Omega = \overline{\mathbb{C}} \backslash K$$

such that $f(\infty) = \infty$ and $f'(\infty) > 0$ and we will call it the Riemann map.

The proof reduces to the original Riemann mapping theorem by use of the inversion $z \mapsto 1/z$.
The quantity $f'(\infty)$ is called the logarithmic capacity of K and is also denoted $\mathrm{cap}(K)$. The denomination "capacity" is justified by the following property :

Proposition 2.2.1. *The quantity $\mathrm{cap}(K)$ is increasing in the sense that if $K_1 \subsetneq K_2$ are two distinct $CCF-$compact then $\mathrm{cap}(K_1) < \mathrm{cap}(K_2)$.*

Proof : Let Ω_j, $j = 1, 2$ be the corresponding domains and $f_j : \Delta \to \Omega_j$ the Riemann maps. Since $\Omega_2 \subset \Omega_1$ we may define $\varphi = f_1^{-1} \circ f_2 : \Delta \to \Delta$ which is holomorphic and fixes ∞. By Schwarz lemma,

$$\lim_{z\to\infty} \frac{\varphi(z)}{z} > 1.$$

But this limit also equals $\mathrm{cap}(K_2)/\mathrm{cap}(K_1)$ and the proposition follows.

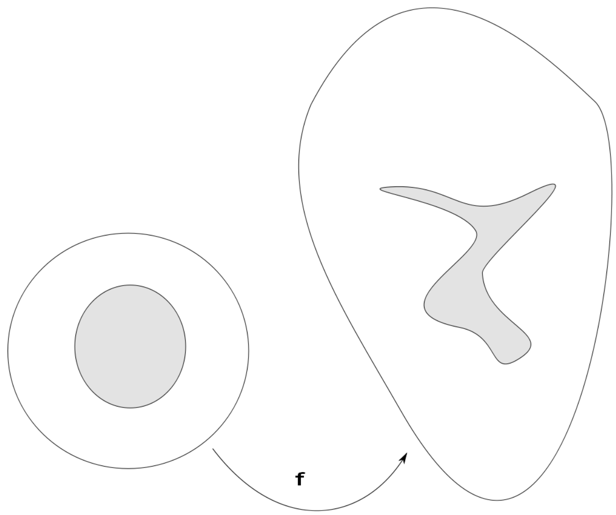


FIG. 2.2 – Riemann map of a CCF-domain

2.3 Around Koebe Theorem

In this section we study the properties of the classes

$$S = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic, injective}; f(0) = 0, f'(0) = 1\}$$

and

$$\Sigma = \{f : \Delta \to \mathbb{C}, \text{ holomorphic and injective; } f(z) = z + b_0 + \frac{b_1}{z} + \text{...at } \infty\}.$$

As an explicit example, we have the *Koebe function*

$$f(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2} = \frac{1}{4}(\frac{1+z}{1-z})^2 - \frac{1}{4}$$

which is the Riemann mapping of $\mathbb{C}\backslash]-\infty, -\frac{1}{4}]$.
We also recall that if K is a $CCF-$compact then the Riemann mapping f of $\Omega = \mathbb{C}\backslash K$ has the development at ∞

$$f(z) = \text{cap}(K)z + b_0 + b_1/z + ...$$

where $\text{cap}(K)$ is the logarithmic capacity. In other words $f \in \Sigma$ if and only $\text{cap}(K) = 1$.

Notice that if $f \in S$ then $\varphi : z \mapsto \frac{1}{f(1/z)} \in \Sigma$. But $f \mapsto \varphi$ is not a bijection between S and Σ ; actually it is a bijection onto the set of functions in Σ that do not vanish.

Theorem 2.3.1. *(Area Theorem) : If cap$(K) = 1$ then ($|.|$ stands for Lebesgue measure)*

$$|K| = \pi(1 - \sum_{n\geq 1} n|b_n|^2).$$

Proof : If γ is a smooth curve surrounding a region A then an immediate application of Stokes formula shows that

$$|A| = \frac{1}{2i}\int_\gamma \overline{z}dz.$$

Let f be the Riemann map of $\overline{\mathbb{C}}\backslash K$. We apply this to $\gamma = f(r\partial\mathbb{D})$ with $r > 1$:

$$\frac{1}{2i}\int_\gamma \overline{z}dz = \frac{1}{2i}\int_0^{2\pi} \overline{f}(re^{i\theta})ire^{i\theta}f'(re^{i\theta})d\theta = \pi(r^2 - \sum_{n=1}^{\infty} n|b_n|^2)$$

and the result follows by letting r decreasing to 1.

Lemma 2.3.1. *: If $f \in S$ then there exists an odd function $h \in S$ such that for $z \in \mathbb{D}$*

$$h(z)^2 = f(z^2).$$

As an example, if f is the Koebe function then h is the Riemann mapping onto the plane minus the two slits $[i/2, +i\infty[$, $[-i/2, -i\infty[$.

Proof : The function $z \mapsto \frac{f(z)}{z}$ does not vanish in $\mathbb{D}$ and thus possesses a square root g. Put $h(z) = zg(z^2)$. it is clearly odd and $h(z)^2 = f(z^2)$. If $h(z_1) = h(z_2)$ then $z_1^2 = z_2^2$ and thus $z_1 = z_2$ since h is odd. Finally $g(z^2) = 1 + .. \Rightarrow h(z) = z + ...$

Theorem 2.3.2. *(Bieberbach) : If $f \in S$ then, if $f(z) = z + a_2z^2 + ..$ we have $|a_2| \leq 2$*

Proof : Let h be as above and

$$g(z) = \frac{1}{h(\frac{1}{z})} = z - \frac{a_2}{2z} + ...$$

An application of the area theorem finishes the proof.

Theorem 2.3.3. *(Koebe) : If $f \in S$ then $f(\mathbb{D}) \supset B(0, \frac{1}{4})$.*

Proof : Let $z_0 \notin f(\mathbb{D})$. The function

$$\tilde{f}(z) = \frac{z_0 f(z)}{z_0 - f(z)} = z + (a_2 + \frac{1}{z_0})z^2 + ..$$

is in S ; by the preceeding theorem $|a_2 + \frac{1}{z_0}| \leq 2$ which in turn implies, since already $|a_2| \leq 2$, that $\frac{1}{|z_0|} \leq 4$.

Corollary 2.3.1. *: If $f : \Omega \to \Omega'$ is holomorphic and bijective and if $f(z) = z', d = d(z, \partial\Omega), d' = d(z, \partial\Omega')$ then*

$$\frac{1}{4}d' \leq d|f'(z)| \leq 4d'.$$

Proof : We may assume $z = z' = 0$. The function $\tilde{f}(w) = \frac{f(dw)}{df'(0)}$ belongs to the class S ; an application of the preceeding theorem shows that $\tilde{f}(\mathbb{D}) \supset B(0, 1/4) \Rightarrow d' \geq \frac{1}{4}|f'(0)|d$.

Theorem 2.3.4. *(Koebe) : If f is holomorphic and injective in the unit disk then for every $z \in \mathbb{D}$*

$$|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}| \leq \frac{4|z|}{1-|z|^2}. \tag{2.2}$$

Proof : Put $T_z(w) = \frac{w+z}{1+\bar{z}w}$. It is an automorphism of $\mathbb{D}$ satisfying $T_z(0) = z$, $T_z'(0) = 1 - |z|^2$. Then

$$\tilde{f}(w) = \frac{f(T_z(w)) - f(z)}{f'(z)(1-|z|^2)} = w + (\frac{f''(z)(1-|z|^2)}{2f'(z)} - \overline{z})w^2 + ..$$

so that $\tilde{f} \in S$. By Koebe's theorem

$$|\frac{f''(z)(1-|z|^2)}{2f'(z)} - \overline{z}| \leq 2$$

and the result follows by multiplication by $\frac{2z}{1-|z|^2}$.

Theorem 2.3.5. *(First distortion Theorem) : If $f \in S$ then, for $z \in \mathbb{D}$*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \tag{2.3}$$

Proof : Put $h = \log f'$. It suffices to prove the theorem for $z = x \in (0, 1)$. We can write

$$x\Re(h'(x)) = \Re(\frac{xf''(x)}{f'(x)})$$

from which it follows, using the preceeding theorem, that

$$\frac{2x-4}{1-x^2} \leq \Re(h'(x)) \leq \frac{4+2x}{1-x^2}$$

and the result follows by integration.

Theorem 2.3.6. *(Second distortion theorem) : If $f \in S$ $z \in \mathbb{D}$ then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \tag{2.4}$$

Proof : The upper bound follows easily from the preceeding theorem :

$$\forall x \in (0,1)\,, |f(x)| \leq \int_0^x \frac{1+t}{(1-t)^3}dt = \frac{x}{(1-x)^3}.$$

The lower bound is obvious if $|f(z)| \geq 1/4$ since $\frac{r}{(1+r)^2}$is always less than $\frac{1}{4}$. Assume now that $f(z) = x \in (0, 1/4)$: then by Koebe theorem $[0, x] \subset \Omega$. Let $C = f^{-1}([0, x]) \subset \mathbb{D}$ then

$$f(z) = \int_C f'(\zeta)d\zeta = \int_C |f'(\zeta)||d\zeta| \geq \int_C \frac{1-|\zeta|}{(1+|\zeta|)^3}|d\zeta| \geq \int_0^x \frac{1-r}{(1+r)^3}dr = \frac{r}{(1+r)^2}.$$

2.4 Capacity

If K is a CCF-compact we have defined the logarithmic capacity of K as

$$\lim_{z\to\infty} \frac{f(z)}{z}$$

where f is the Riemann mapping of $\mathbb{C}\backslash K$.

Proposition 2.4.1. *: The logarithmic capacity is translation and rotation invariant. If $h(z) = \lambda z$ then* $\text{cap}(h(K)) = |\lambda|\text{cap}(K)$.

The easy proof is left to the reader.
As a consequence we observe that the (logarithmic) capacity of a disc of radius r is equal to r and since the map $z \mapsto z + 1/z$ sends Δ onto $\mathbb{C}\backslash[-2, 2]$ we see that the capacity of a line segment of length l is $l/4$.
We have already proven the monotonicity of logarithmic capacity : If K, K' are full compact sets and $K \subset K'$ then $\text{cap}(K) \subset \text{cap}(K')$ with equality if and only if $K = K'$.

Proposition 2.4.2. *: If K is a $CCF-set$ then* $\text{cap}(K) \leq \text{diam}(K) \leq 4\text{cap}(K)$.

Proof : We assume again that $0 \in K$ and define $\varphi(w) = 1/f(1/w)$. Computation shows that $\varphi'(0) = 1/\text{cap}(K)$. The rest follows from Koebe theorems.

2.4.1 Equilibrium measures

We now wish to generalize the notion of capacity to general compact subsets of the plane. If $E \subset \mathbb{C}$ is compact, μ is a finite Borel measure on E we define its potential by

$$P_\mu(z) = \int_E \log|z - \zeta| d\mu(\zeta).$$

This is well defined since the integrand is the sum of a negative function and a bounded one. It is harmonic outside E and, by Fatou lemma, upper-semicontinuous,

i.e. a subharmonic function on the whole plane.
The energy of a finite Borel measure on E is $I_\mu = \int P_\mu d\mu$. An equilibrium state is a probability measure on E that maximizes the energy among all probability measures. A set will be called polar if $I(\mu) = -\infty$ for all probability measures with support included in the set.

Theorem 2.4.1. *If E is non polar, it supports an equilibrium measure with finite energy.*

Lemma 2.4.1. *If (μ_n) is a sequence of probability measures on E converging weakly to a probability measure μ then*

$$I(\mu) \geq \overline{\lim} I(\mu_n).$$

Proof : This follows from the fact that

$$\overline{\lim}\, I(\mu_n) \leq \overline{\lim} \int \int \max\left(\log|z-\zeta|, -m\right) d\mu_n(z) d\mu_n(\zeta).$$

We pass then to the weak limit on the right-hand side using the fact that $\mu_n \otimes \mu_n$ converges weakly to $\mu \otimes \mu$. We then let $m \to \infty$ and use monotone convergence theorem.
To finish the proof of the theorem it suffices to take a weak limit of a subsequence of (μ_n) where $I(\mu_n)$ converges to the supremum.
The following theorem describes the potential of an equilibrium measure.

Theorem 2.4.2. *(Frostman) Let K be a non-polar compact set and ν an equilibrium measure for K. Then $P_\nu \geq I(\nu)$ everywhere while $P_\nu = I(\nu)$ on $K \backslash L$ where L is polar.*

Proof : We first show that $K_\varepsilon = \{z \in K\,;\, P_\nu(z) \geq I(\nu) + \varepsilon\}$ is polar if $\varepsilon > 0$. If not, choose a probability measure μ on K_ε such that $I(\mu) > -\infty$. Because $I(\nu) = \int P_\nu d\nu$, there exists a $z \in K$ such that $P_\nu(z) \leq I(\nu) + \varepsilon/4$. By upper-semi-continuity, $P_\nu(z) \leq I(\nu) + \varepsilon/2$ on $D(z,r)$ which must be disjoint from K_ε. Let $a = \nu(\overline{D}(z,r))$ and define the measure σ as being equal to μ on K_ε, $-\nu/a$ on $\overline{D}(z,r)$ and 0 elsewhere. The measure $\nu_t = \nu + t\sigma$ is then positive for $t < a$ and an easy calculation shows that $I(\nu_t) > I(\nu)$ if $t > 0$ is small enough. It follows that $P_\nu \leq I(\nu)$ except on a polar set in K. The second part of the proof starts with a

Lemma 2.4.2. *If $E \subset K$ is polar then $\mu(E) = 0$ for all measures μ such that $I(\mu) > -\infty$.*

Proof : We may assume E compact : if $\mu(E) > 0$ then it is easy to see that $I(\mu|E) > 0$, contradicting E polar.

So we have $P_\nu \leq I(\nu)$ ν-almost everywhere. By upper-semicontinuity, if P_ν were $< I(\nu)$ somewhere on the support of ν, it would remain true on a disk centered at this point, thus of positive ν measure, contradicting the definition of equilibrium measure. Thus $P(\nu) \geq I(\nu)$ on the support of ν and hence everywhere by the minimum principle and the first part of the proof shows that we have actually equality outside a countable union of polar compact subsets of K.
We define for a compact set E cap(E) as being equal to 0 if E is polar and to be equal to $e^{I(\nu)}$ otherwise. By the definition of energy this quantity is increasing wrt E. In the next paragraph we show that this notion coincides with logarithmic capacity in the connected case.

2.4.2 Green's function

Let U be a proper sudomain of the plane.

Definition 2.4.1. *A Green's function for U is a map $g_U : U \times U \to [0, +\infty[$ satisfying*
(1) $g_U(., w)$ is harmonic in $U \backslash \{w\}$ for every $w \in U$ and bounded outside any neighborhood of w.
(2) $g_U(w, w) = +\infty$ and , as $z \to w$, $g_U(z, w) = -\log|z - w| + O(1)$ if $w \neq \infty$ and $g_U(z, \infty) = \log|z| + O(1)$ in the case $w = \infty$.
(3) For fixed $w \in U$, $g_U(z, w) \to 0$ as $z \to \zeta \in \partial U$ nearly everywhere on ∂U, meanning for all but a countable union of polar compact subsets of ∂U.

Let U be simply connected : denote for each $w \in U$, Φ_w the Riemann mapping sending 0 to w.Then $g_U(z, w) = -\log|\Phi_w^{-1}(z)|$ is a Green's function for U.

Theorem 2.4.3. *If ∂U is non polar there exists a unique Green's function for U.*

Proof : 1) Uniqueness : w being fixed, if g_1, g_2 are two Green's functions define $h(z) = g_1(z, w) - g_2(z, w)$. This is a harmonic function on U which is bounded with boundary values 0. It must be identically 0 by maximum principle.
2) Existence : We assume first $\infty \in U$. Let ν be the equilibrium measure for ∂U. We define $g_U(z, \infty) = P_\nu(z) - I(\nu)$ which is easily seen to satisfy all the requirements. For a finite point w, let $U' = T(U)$ where $T(z) = \frac{1}{z-w}$ and define $g_U(z, w) = g_{U'}(T(z), \infty)$.

The above proof shows that if f denotes the Riemann map of the $CCF-$domain $\Omega = \mathbb{C} \backslash K$ then

$$\log|f^{-1}(z)| = P_\nu(z) - I_\nu(z),$$

where ν is an equilibrium measure, from which it follows that

$$\text{cap}(K) = e^{I(\nu)}$$

as desired.

2.5 Harmonic measure and Beurling theorem

If Ω is a bounded plane domain, we say that Ω is regular for the Dirichlet problem if one can solve Dirichlet problem in Ω for every function f continuous on $\partial\Omega$, i.e. one can find a function u continuous on $\overline{\Omega}$ such that $u|\Omega$ is harmonic and such that $u|\partial\Omega = f$. The maximum principle implies that, if $z \in \Omega$ the application $f \mapsto u(z)$ is a continuous linear form on the space $C(\partial\Omega)$. By the Riesz representation theorem, there exists a unique probability measure μ on $\partial\Omega$ such that for any $f \in C(\partial\Omega), u(z) = \int_{\partial\Omega} f d\mu$.

Definition 2.5.1. *This measure is called the harmonic measure at point z in Ω, and written as $\omega(z, \Omega, .)$ or ω^z if the context is clear.*

Using inversion about 0 we can define as well harmonic measure for (regular) domains whose complement is a compact set, i.e. domains that contain ∞. For these domains, there is a nice characterization of harmonic measure at ∞ :

Theorem 2.5.1. *Let Ω be a regular domain containing ∞ and K its boundary. Then the equilibrium measure for K is precisely ω^∞.*

Proof : The function $u(z) = P_\omega(z) - P_\nu(z) + I(\nu)$ extended by $u(\infty) = I(\nu)$ is harmonic in Ω and converges to $P_\omega(\zeta)$ at every boundary point. It follows that $u(z) = \int_{\partial\Omega} P_\omega(\zeta) d\omega^z(\zeta)$. Specializing $z = \infty$ it follows that $I(\nu) \leq I(\omega)$ and thus that $\nu = \omega$ because of the

Proposition 2.5.1. *A non-polar compact set admits a unique equilibrium measure.*

Proof : This follows immediately from the fact that $\Delta P_\mu = \mu$ in the sense of distributions for every probability measure μ.

There is an equivalent probalistic definition of harmonic measure. Let B_t^z be a standard Brownian motion started at point z and let $\tau_\Omega = \inf\{t \geq 0; B_t \notin \Omega\}$. Then, for a Borel subset E of $\partial\Omega$,

$$\omega(z, \Omega; E) = P[B^z_{\tau_\Omega} \in E]$$

and this formula implies the more general, and useful following one :

$$\forall f \in C(\partial\Omega), u(z) = E[f(B^z_{\tau_\Omega})].$$

2.5.1 Beurling theorem

We now investigate the size of the harmonic measure of a set E, in terms of its diameter and distance to z.

Theorem 2.5.2. *Let λ be a full continuum joining 0 and the boundary of the unit disk. Then if $z \in \mathbb{D}\backslash\lambda$ then the probability that a Brownian motion starting from z will hit the boundary of the circle before hitting λ is smaller than $c\sqrt{|z|}$ In other terms,*

$$\omega(z, \mathbb{D}\backslash\lambda; \partial\mathbb{D}) \leq c\sqrt{|z|}.$$

Here is an intuitive proof of this result : if $|z| = r$ then this probability is $\geq$ than the same probability in the case $z = -r, \lambda = [0, 1]$ where the result follows by an explicit computation.
This intuitive idea is made rigourous in the following theorem, due to Beurling :

Theorem 2.5.3. *Let α be a compact subset of the unit disk $\mathbb{D}$ and α^* its circular projection on the negative radius, i.e.*

$$\alpha^* = \{-r; \exists z \in \alpha; |z| = r\}.$$

We make the further hypothesis that $\mathbb{D}\backslash\alpha$ and $\mathbb{D}\backslash\alpha^*$ are regular for the Dirichlet problem. If $x \in (0, 1)$ then $\omega(x) \geq \omega^*(x)$ where $\omega^*(z) = \omega(z, \mathbb{D}, \alpha^*)$.

Proof : Let $g(z, w) = \log|\frac{1-\bar{z}w}{z-w}|$ be the Green's function for the unit disk. We leave to the reader the proof of the following elementary inequalities :

$$g(|z|, -|w|) \leq g(z, w) \leq g(|z|, |w|) \leq g(-|z|, -|w|).$$

By Green's formula,

$$\omega^*(z) = -\frac{1}{\pi}\int_{\alpha^*} g(\zeta, z)\frac{\partial\omega^*(\zeta)}{\partial n}|d\zeta| \ . \tag{2.5}$$

In the formula (2.5) we can replace each $\zeta \in \alpha^*$ by $\zeta' \in \alpha$ such that $|\zeta'| = |\zeta|$ and in such a way that the function $\zeta' \to g(\zeta', z)$ is measurable. We may then define the function

$$u(z) = -\frac{1}{\pi}\int_{\alpha^*} g(\zeta', z)\frac{\partial\omega^*(\zeta)}{\partial n}|d\zeta|.$$

It is harmonic outside α and it vanishes on $\partial\mathbb{D}$. Moreover by the inequalities above, $u(z) \leq \omega^*(z) \leq 1$, $z \in \mathbb{D}\backslash\alpha$, and thus $u(z) \leq \omega(z)$. On the other hand $u(x) \geq \omega^*(x)$ and the result follows.

Corollary 2.5.1. *: Let Ω be a simply connected domain with $\infty \in \partial\Omega$, $z_0 \in \partial\Omega$ and $z \in \Omega; \mathrm{d}(z, \partial\Omega) > r$: then the probability that a Brownian motion starting from z will hit $\partial B(z_0, r)$ before $\partial\Omega\backslash B(z_0, r)$ is smaller than $c\sqrt{\frac{r}{|z-z_0|}}$.*

The corollary follows from the theorem by an inversion about z_0.

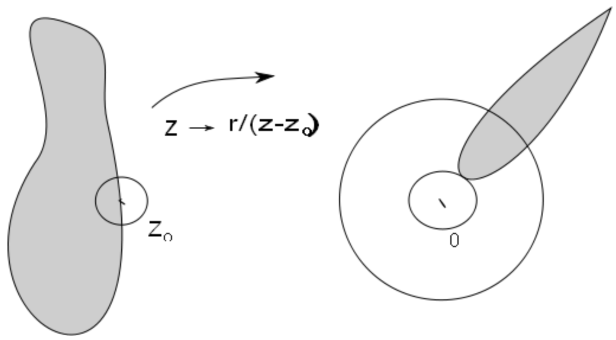


FIG. 2.3 – Beurling's estimate

2.6 Caratheodory convergence theorem

let (Ω_n) be a sequence of $CCF-$domains and (f_n) the associate sequence of Riemann maps. We recall that, Ω_n being a $CCF-$domain, $0 \notin \Omega_n$. The aim of the present paragraph is to characterize geometrically the analytic property that (f_n) converges uniformly on compact subsets of Δ. First of all we need to precise what we mean by uniform convergence on a compact neighborhood of ∞ : since all functions f_n avoid 0, this is the same as asking that the sequence

$$z \mapsto \frac{1}{f_n(1/z)}$$

converges uniformly on compact subsets of the unit disk. Let $K_n = \overline{\mathbb{C}} \backslash \Omega_n$: if the sequence $\mathrm{cap}(K_n)$ is bounded from below then, by Koebe distortion theorem, uniform convergence on compact subsets of Δ is the same as uniform convergence on compact subsets of $\Delta \backslash \infty$.

Definition 2.6.1. *The kernel of a sequence (Ω_n) of $CCF-$domains is the union of all domains $U \subset \overline{\mathbb{C}}$ such that $\infty \in U$ and $U \subset \Omega_n$ for n large enough. If no such domain exists we say that the kernel is $\{\infty\}$.*

Exemples :
1) If the sequence (Ω_n) is increasing its kernel is $\bigcup_{n \geq 0} \Omega_n$.
2) If the sequence (Ω_n) is decreasing let $E = \bigcap_{n \geq 0} \Omega_n$: if E is a neighborhood of ∞ then the kernel of E is the connected component of $\overset{\circ}{E}$ containing ∞, otherwise the kernel is $\{\infty\}$.

Definition 2.6.2. *We say that the sequence (Ω_n) is convergent if every subsequence has the same kernel Ω. We then say that (Ω_n) converges in the sense of Caratheodory to Ω.*

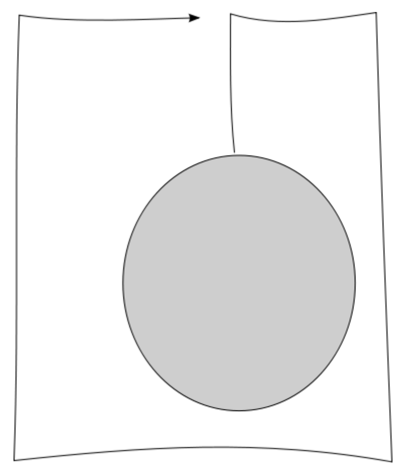


FIG. 2.4 – The kernel is the complement of the square

Exemples : Monotone families converge in the sense of Caratheodory.
As a consequence every decreasing one-parameter family of $CCF-$domains has a limit from the left and from the right at every $t \geq 0$. For this statement to make sense we precise that we say that (Ω_t) converges in the sense of Caratheodory to Ω as $t \to t_0$ if for every sequence $t_n \to t$ the sequence (Ω_{t_n}) converges in the sense of Caratheodory to Ω.

Theorem 2.6.1. *Let (Ω_n) be a sequence of $CCF-$domains and (f_n) the corresponding sequence of Riemann maps. Then the sequence (f_n) is uniformly convergent on compact subsets of Δ if and only if (Ω_n) converges in the sense of Caratheodory to a kernel distinct from $\overline{\mathbb{C}}\backslash\{0\}$. If (Ω_n) converges and if Ω denotes its kernel then*
a) If $\Omega = \{\infty\}$, $f_n \to \infty$ uniformly on compact subsets of Δ.
b) If $\Omega = \overline{\mathbb{C}}\backslash\{0\}$ f_n converges to 0 uniformly on compact subsets of $\Delta\backslash\{\infty\}$.
c) Otherwise f_n converges to f, the Riemann mapping of Ω.

Proof :
1) Assume first that f_n converges to f uniformly on compact subsets of Δ and let us prove that $f(\Delta)$ is the kernel of (Ω_n).
a) Let $w_o \in f(\Delta), w_o \neq \infty$. We must show that there exists a domain U containing ∞, w_o such that $U \subset f_n(\Delta)$ for n large enough. Choose $r > 1$ such that $1 < r < |z_o|$ where $f(z_o) = w_o$ and define $U = f(\{|z| > r\})$. Arguing by contradiction, assume that U does not satisfy the required property. Then there exists $n_k \to \infty$, $w_k \in U$ such that $w_k \notin f_{n_k}(U)$. WLOG we may assume that

$$w_k \to w \in \overline{U} \subset f(\Delta) \ \ (*).$$

But the functions

$$z \mapsto f_{n_k}(z) - w_k$$

do not vanish in Δ. Since f is not constant the function $z \mapsto f(z) - w$ does not vanish either, in contadiction with (*). It follows that $f(\Delta) \subset \text{kernel}(\Omega_n)$.
b) Let now w_o belong to the kernel of (Ω_n) : there thus exists a domain U containing w_o, ∞ such that $U \subset \Omega_n$ for n large enough. Put $g_n = f_n^{-1} : \Omega_n \to \Delta$: this defines a normal family, having thus a convergent subsequence on compact subsets of U that we still denote $(g_n), g$ denoting the limit. Since g is open we have $|g(w_o)| > 1$. Now since $g_n(w_o) \to g(w_o)$ and $w_o = f_n(g_n(w_o))$ we have $w_o = f(g(w_o)) \in f(\Delta)$, which ends the first part of the theorem.
2) Suppose that (Ω_n) converges to its kernel Ω. If there existed a subsequence $n_k \to \infty$ such that $\text{cap}(K_{n_k}) \to 0$ then Koebe theorem would imply, since $\{|z| > 4\text{cap}(K_{n_k})\} \subset \Omega_n$, that the kernel is $\overline{\mathbb{C}} \backslash \{0\}$, which was excluded. So the sequence $(\text{cap}(K_n))$ is bounded from below, implying that (f_n) is a normal family. To finish, it remains to prove that cannot have two subsequences converging to different limits : but this follows from the first part of the proof.

2.7 Boundary Behaviour

In the preceeding section we have seen that any two $CCF-$domains are conformally equivalent, in the sense that there exists an holomorphic homeomorphism between the two domains. In this section we examine the question of the boundary behaviour of this conformal homeomorphism.
Before we study in details this problem let us first notice an easy but useful result :

Proposition 2.7.1. *If U, V are two subdomains of the Riemann sphere and if $f : U \to V$ is an homeomorphism then, if $z_n \in U$ is a sequence converging to ∂U, every limit value of the sequence $f(z_n)$ belongs to ∂V.*

Proof : We may assume WLOG that the sequence $(f(z_n))$ converges to v : if $v \in V$ then $z_n = f^{-1}(f(z_n))$ converges to $f^{-1}(v)$, a contradiction.

The important concept for this paragraph is the notion of cross-cut :

Definition 2.7.1. *A crosscut Γ in a domain Ω is an open Jordan arc such that $\overline{\Gamma} = \Gamma \cup \{a, b\}$, $a, b \in \partial\Omega$.*

Proposition 2.7.2. *If C is a crosscut of the CCF-domain Ω then $\Omega \backslash C$ has exactly two components.*

Proof : Let $H(z) = z(|z| - 1)$, so that H is an homeomorphism from Δ onto $\overline{\mathbb{C}} \backslash \{0\}$. Let $g = f^{-1}$ where f is the Riemann mapping of Ω : then $H \circ g$ is an homeomorphism from Ω onto $\overline{\mathbb{C}} \backslash \{0\}$ sending the crosscut C onto a Jordan curve

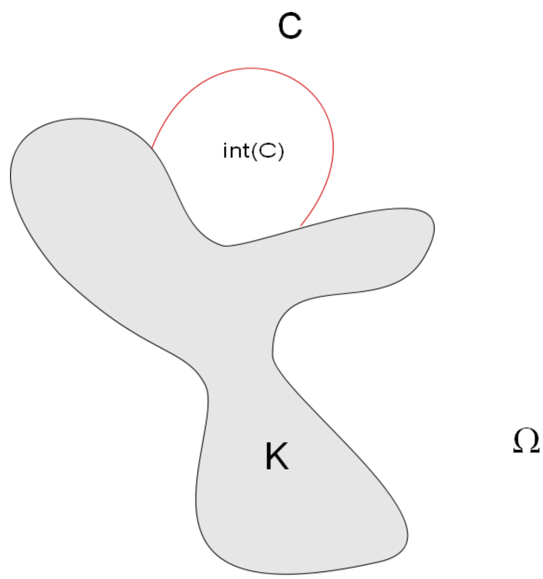


FIG. 2.5 – Crosscut

of the Riemann sphere containing 0 by proposition (2.7.1). The proposition then follows from the Jordan curve theorem.

Definition 2.7.2. *If C is a crosscut of a $CCF-$domain Ω, we denote by* $\mathrm{int}(C)$ *the connected component of the complement of C in Ω that does not contain ∞.*

We consider now two $CCF-$domains Ω, Ω', f the holomorphic bijection between Ω and Ω' such that $f'(\infty) > 0$ and a point $w \in \partial\Omega$. The following lemma is a technical tool that will be used intensively in the rest of the paragraph.

Lemma 2.7.1. *There exists a sequence (r_n) converging to 0 such that $l(f(C(r_n)\cap \Omega) \to 0$ where l denotes length and $C(r) = \partial D(w,r)$.*

Proof : Put $l(r) = l(f(C(r)\cap\Omega))$. By Cauchy-Schwarz inequality,

$$l(r)^2 \leq 2\pi r \int_{t;w+re^{it}\in\Omega} |f'(w+re^{it})|^2 r dt,$$

so that

$$\int_0^{r_0} \frac{l(r)^2}{r} dr \leq 2\pi A.$$

This inequality implies that

$$\frac{1}{2\pi}\int_0^1 \frac{l(r)^2}{r} dr$$

is less or equal than the area of the image by f of $\{z\in\Omega;\, \mathrm{dist}(z,\partial\Omega) < 1\}$ which is finite, and the lemma follows.
The following theorem is a first example of application of the lemma :

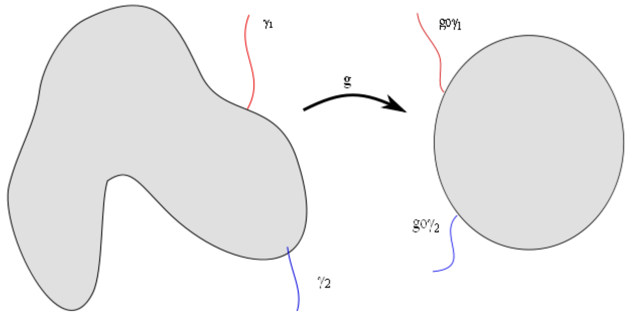


FIG. 2.6 – Theorem 2.7.1

Theorem 2.7.1. *Let Ω be a $CCF-$domain, $f : \Delta \to \Omega$ its Riemann map and $g = f^{-1}$. If $\gamma : [0,1] \to \mathbb{C}$ is a curve such that $\gamma(0) \in \partial\Omega$, $\gamma(]0,1]) \subset \Omega$, then $g \circ \gamma$, which is defined on $(0,1]$, has a continuous extention at 0 and $g \circ \gamma(0) \in \partial\Delta$. Moreover if we consider two such curves $\gamma_j, j = 1,2$ such that $\gamma_1(0) \neq \gamma_2(0)$ then $g \circ \gamma_1(0) \neq g \circ \gamma_2(0)$.*

Proof : For $0 < r < 1$ we consider the open disks $D(r) = D(\gamma(0), r)$ and denote by $C(r)$ their boundary. Let $U(r)$ be the connected component of $D(r) \cap \Omega$ containing $\gamma(t)$ for t small, and $\delta(r) = \partial U(r) \cap \Omega$. By the lemma, there exists a sequence $r_n \to 0$ such that $g(\delta(r_n))$ is a sequence of crosscuts of Δ whose diameters converge to 0. But then the diameters of $g(U(r_n))$ also converge to 0 and the first part of the theorem follows. To prove the second part we argue by contradiction and consider $r > 0$ small enough so that $\overline{U_1(r)}$ does not contain $\gamma_2(0)$: this is impossible since $g(U_1(r))$ must contain $g(U_2(r'))$ for some small r'.

In this last proof we have crucially used the fact that $\mathrm{diam}(g(\delta(r)))$ small $\Rightarrow$ $\mathrm{diam}(g(U(r)))$ is also small. This comes from a property of $\partial\Delta$. If we replace Δ by any domain having such property, the theorem should extend. This is precisely the way we are going to prove Caratheodory's theorem characterizing the CCF-domains Ω for which f ,its Riemann mapping, extends continuously to $\overline{\Delta}$.

Definition 2.7.3. *A compact set $X \subset \mathbb{C}$ is said to be locally connected if*

$$\forall\varepsilon > 0 \exists\delta > 0; \forall x, y \in X, |x-y| < \delta \Rightarrow \exists X_1 \subset X \text{connected}; x, y \in X_1, \mathrm{diam}(X_1) \leq \varepsilon.$$

Theorem 2.7.2. *(Caratheodory) : The mapping f has a continuous extension to $\overline{\Delta}$ if and only if $\partial\Omega$ is locally connected.*

Proof : It is easy to see that the continuous image of a locally connected compact set is again compact and locally connected so the only if part will follow from the fact that if f extends continuoulsly then $\partial\Omega = f(\partial\Delta)$. To prove this last fact consider first $z \in \partial\Omega$: this point is the limit of a sequence (z_n) of points in Ω . But $z_n = f(\omega_n)$ for a sequence $(\omega_n) \in \Delta$ and wlog we may assume that

$\omega_n \to \omega \in \partial\Delta$, from which it follows that $z = f(\omega)$. For the other inclusion suppose that there exists $x \in \partial\Delta$ such that $f(x) \in \Omega$; then there must exist $\omega \in \Delta$ such that $f(x) = f(\omega)$ and , if we denote by γ a curve joining ω to x in Δ , $f(\gamma)$ is a compact subset of Ω . But this is impossible since then $\gamma = g(f(\gamma))$ must be compact in Δ.
We come to the converse. By the lemma we can find for every $z \in \partial\Delta$ a sequence r_n converging to 0 such that $\gamma_n = f(\Delta \cap \partial D(z, r_n))$ is a crosscut in Ω of diameter converging to 0 and whose endpoints a_n, b_n converge to a point $\omega \in \partial\Omega$. By the local connectedness assumption there exists a connected subset of $\partial\Omega$ containing a_n, b_n , say L_n, with $\mathrm{diam}(L_n) = \varepsilon_n \to 0$. If $w \in \Omega$, $|w - a_n| > \varepsilon_n$ and if the same is true for z_0 then these two points are separated neither by $C_n \cup L_n$ nor by $\mathbb{C}\backslash\Omega$. We then invoke the following...

Theorem 2.7.3. *(Janiszewski) If A, B are closed sets of the complex plane such that $A \cap B$ is connected then, if a, b are two points of the plane which are separated neither by A nor B, then they are not separated by $A \cup B$.*

...and conclude that w and z_0 are not separated by $C_n \cup L_n \cup (\mathbb{C}\backslash\Omega) = C_n \cup \mathbb{C}\backslash\Omega$. It follows that $U_n \subset \{|w - a_n| \leq \varepsilon_n\}$ and consequently that $\mathrm{diam}(U_n) \to 0$. Continuity of f at the point z then easily follows.

We will often deal with $CCF-$domains Ω of the form $\Delta\backslash\gamma([0, t])$ where γ is continuous and injective from $[0, t]$ into $\overline{\Delta}$ with $\gamma(0) \in \partial\Delta$ and $\gamma(]0, t]) \subset \Delta$. These domains have obviously a locally-connected boundary so that the Riemann mapping $f : \Delta \to \Omega$ has a continuous extension to $\overline{\Delta}$. But the reciprocal map $g = f^{-1}$ cannot have a continuous extension to $\overline{\Omega}$ since the points $\gamma(x)$, $0 < x < t$ have two preimages by f. Nevertheless we can say something about the point $\gamma(t)$.

Definition 2.7.4. *A point $\omega \in \partial\Omega$ is called a cut point if $\partial\Omega\backslash\{\omega\}$ is not connected.*

Lemma 2.7.2. *Let Ω be a $CCF-$domain and f be its Riemann map. Assume $\partial\Omega$ is locally connected : then f assumes the value $\omega \in \partial\Omega$ exactly once if and only if ω is not a cut point of $\partial\Omega$.*

Proof : If a is the only preimage of ω then $\partial\Omega\backslash\{\omega\} = f(\partial\Delta\backslash\{a\})$ is connected. Conversely assume that $f(a) = f(a') = \omega$. Let l be a crosscut from a to a' in Δ : then $f(l) \cup \{\omega\} = \Gamma$ is a Jordan curve. By Jordan curve theorem its complement consists of two open components U_1, U_2 and $\partial\Omega\backslash\{\omega\} = (\partial\Omega \cap U_1) \cup (\partial\Omega \cap U_2)$. Now let V be the interior part of l : $f(V)$ must lie inside the bounded component of the complement of Γ, say U_1. Let F be an holomorphic function in Δ with a continuous extension to $\overline{\Delta}$: it is known that then $\log|F|$ is integrable on $\partial\Delta$, from which it follows that F cannot vanish on an interval. Applying this to $F = f - \omega$ we see that if I is the closed circular arc joining a, a' and included in ∂V, then $f(I)$ must contain a point different from ω and this point must belong to $\partial\Omega \cap U_1$.

On the other hand $f(\partial\Delta\backslash I)$ must be included in $\partial\Omega \cap U_2$. The set $\partial\Omega\backslash\{\omega\}$ thus cannot be connected, being a union of two relatively open nonvoid sets.

As a corollary we have the following precised Caratheodory theorem for Jordan domains :

Theorem 2.7.4. *: The domain Ω is a Jordan domain if and only if f extends to a homeomorphism from $\overline{\Delta}$ to $\overline{\Omega}$.*

Corollary 2.7.1. *: If $\Omega_j, j = 1, 2$ are two Jordan domains then every holomorphic bijection between the two domains extends to a homeomorphism of the closures. Moreover, fixing $z_j, j = 1, 2, 3$ in this order in the trigonometric sense on $\partial\Omega_1$ and simlarly $z'_j, j = 1, 2, 3$ on $\partial\Omega_2$ there is a unique holomorphic bijection between Ω_1 and Ω_2 whose extention sends z_j to $z'_j, j = 1, 2, 3$.*

Proof : Using Riemann mapping theorem and the last one it suffices to prove the corollary for $\Omega_j = \mathbb{D}, j = 1, 2$ where the result follows from the fact that an automorphism of the disk depends on three (real) parameters.

Chapitre 3

Löwner Differential Equation

3.1 Radial Löwner Processes

3.1.1 Definition and first properties

Let $(K_t)_{t\geq 0}$ be a (strictly) increasing family of $CCF-$ sets, i.e. a growing family of full connected compact sets containing 0. We denote by (Ω_t) the complement of K_t and by f_t the Riemann map of Ω_t, i.e. the unique holomorphic bijection from Δ onto Ω_t such that

$$f_t(\infty) = \infty \text{ and } \lim_{z\to\infty} \frac{f_t(z)}{z} > 0.$$

We may then write $f_t(z) = c(t)z + ...$ where $c(t) = \text{cap} K_t$ is the logarithmic capacity.
We make the following assumptions :
1) The family (Ω_t) is continuous in the sense of Caratheodory convergence.
As we have seen this is equivalent to saying that

$$\forall t_o \geq 0,\ \Omega_{t_o} = \cup_{t>t_o}\Omega_t$$

and

$$\forall t_o > 0,\ \Omega_{t_o} \text{is the component of the interior of} \cap_{t<t_o} \Omega_t \text{ containing} \infty.$$

This property is equivalent, as we have seen, to the fact that the family (f_t) is continuous in t for the topology of uniform convergence on compact sets, and in particular the function $t \mapsto c(t)$ is continuous and stricly increasing.
2) $\lim_{t\to\infty} c(t) = +\infty$ and $c(0) = 1$.
Frequently we will assume that $K_0 = \overline{\mathbb{D}}$.
If these conditions are satisfied one may perform a time-change and assume that $c(t) = e^t$.

We start the study of such growth processes by observing that if $s \leq t$, $f_t(\Delta) \subset f_s(\Delta)$ so that $h_{s,t}(z) = f_s^{-1} \circ f_t$ is a well-defined map from Δ into itself fixing ∞. It is easy to see that if $s_n \nearrow t$ then $h_{s_n,t} \to id$ uniformly on compact sets and the same is true for h_{s,t_n} if $t_n \searrow s$. From this it follows easily that if (K_t) is a growing family of $CCF-$sets such that $t \mapsto \text{cap}(K_t)$ is continuous, then (Ω_t) is Caratheodory continuous.

Definition 3.1.1. *If $f, g : \Delta \to \mathbb{C}$, are two holomorphic functions we say that f is subordinate to g (and denote this by $f \prec g$) if there exists $\varphi : \Delta \to \Delta$ holomorphic and fixing ∞ such that $f = g \circ \varphi$.*

Notice that, by Schwarz lemma, $|\varphi(z)| \geq |z|$ so that not only $f(\Delta) \subset g(\Delta)$ but also, for every $r > 1$, $f(\{|z| > r\}) \subset g(\{|z| > r\})$.

Definition 3.1.2. *The family $(f_t)_{t \geq 0}$ of holomorphic and injective mappings from Δ into $\mathbb{C}$ is called a Löwner chain if*
1) $f_t(z) = e^t z + ...,$
2) $f_t \prec f_s$ if $0 \leq s \leq t$.

We have already seen that if (f_t) is a Löwner chain then it has to be continuous for the topology of uniform convergence on compact sets. The next proposition is a considerable strengthening of this statement, since it shows that we may replace continuity by absolute continuity.

Proposition 3.1.1. *If (f_t) is a Löwner chain then for $0 \leq s \leq t$,*

$$\forall z \in \Delta,\, |f_t(z) - f_s(z)| \leq 128(e^t - e^s)|z|^3 \frac{(|z|+1)^2}{(|z|-1)^4}.$$

In particular for any $z \in \Delta$, $t \mapsto f_t(z)$ is absolutely continuous (AC) with respect to Lebsgue measure on $\mathbb{R}$.

Proof : There exists a curve γ joining z to $h_{s,t}(z)$ inside $\{|\zeta| \geq |z|\}$ with length $\leq 2|h_{s,t}(z) - z|$. Then, writing $f_t(z) = f_s(h_{s,t}(z))$ we have the inequality

$$|f_t(z) - f_s(z)| \leq 2|h_{s,t}(z) - z| \sup_{|u| \geq |z|} |f_s'(u)|.$$

Lemma 3.1.1. *Let $\Phi : \Delta \to \mathbb{C}$ be a Riemann mapping of a domain not containing 0, $\Phi(z) = cz + .., c > 0$, then $|\Phi(z)| \leq 4c|z|$, $z \in \Delta$.*

Proof : It merely consists in applying the second distortion theorem to the map $\varphi(\omega) = c/\Phi(1/\omega)$ which belongs to the class S.
If $\Phi = f_s$ we write F_s for the corresponding φ we get

$$\forall u \in \Delta\ , |f_s'(u)| \leq |F_s'(1/u)||f_s(u)|^2 e^{-s}|u|^{-2}.$$

Applying the lemma together with the first distortion theorem for F_s we obtain

$$|f_s'(u)| \leq 16e^s|u|^2\frac{(|u|+1)}{(|u|-1)^3}.$$

Similarly, a direct application of the lemma to the function $h_{s,t}$ gives

$$|h_{s,t}(z)| \leq 4e^{t-s}|z|.$$

We now come to the estimation of $|h_{s,t}(z) - z|$. First of all, by Schwarz lemma, $|h_{s,t}(z)| \geq |z|$, $z \in \Delta$. It follows that the function defined by

$$p_t(z) = \frac{e^{t-s}+1}{e^{t-s}-1}\frac{h_{s,t}(z)-z}{h_{s,t}(z)+z}$$

belongs to the class $\mathcal{P}(\Delta)$ of holomorphic functions in Δ with value 1 at ∞ and with positive real part.

Lemma 3.1.2. *If $p \in \mathcal{P}(\Delta)$ then*

$$\frac{|z|-1}{|z|+1} \leq |p(z)| \leq \frac{|z|+1}{|z|-1}$$

and this shows in particular that $\mathcal{P}(\Delta)$ is a normal family.

Proof : The mapping

$$\zeta \mapsto \frac{\zeta+1}{\zeta-1}$$

maps $\{x > 0\}$ onto Δ. Thus the map

$$z \mapsto \frac{p(z)+1}{p(z)-1}$$

sends Δ into itself and fixes ∞. By Schwarz lemma

$$\frac{p(z)+1}{p(z)-1} = u$$

with $|u| \geq |z|$. Since then

$$p(z) = \frac{1+u}{u-1}$$

we must have

$$\frac{|z|-1}{|z|+1} = \inf_{|u|\geq|z|}|\frac{u+1}{u-1}| \leq |p(z)| \leq \sup_{|u|\geq|z|}\left|\frac{u+1}{u-1}\right| = \frac{|z|+1}{|z|-1}.$$

Applying the last lemma we get

$$|h_{s,t}(z)-z| \leq \frac{(e^{t-s}-1)}{(e^{t-s}+1)}|h_{s,t}(z)+z|\frac{|z|+1}{|z|-1} \leq 4(e^{t-s}-1)|z|\frac{|z|+1}{|z|-1}.$$

Combining all the estimates we can conclude the proof of the proposition.

3.1.2 Löwner differential Equation.

We come to the heart of the matter :

Theorem 3.1.1. *Let $(f_t)_{t\geq 0} : \Delta \to \mathbb{C}$ a family of holomorphic functions with $f_t(\infty) = \infty$.If the family (f_t) is a Löwner chain the two following conditions are satisfied*
(1) For each $z \in \Delta\,, t \mapsto f_t(z)$ is absolutely continuous. Moreover, f_0 is injective in Δ and $\forall t \geq 0,\ f_t(z) = e^t z + ..$ at ∞.
(2) There exists a measurable family (p_t) of functions in $\mathcal{P}(\Delta)$, a borelian subset of full measure of $[0, +\infty[$ such that for all $z \in \Delta$, $t \mapsto f_t(z)$ is differentiable in E and ,

$$\forall z \in \Delta\,, \forall t \in E,\ \frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z} p_t(z). \tag{3.1}$$

Conversely, given a measurable family (p_t) of functions in $\mathcal{P}(\Delta)$ and a function φ holomorphic in Δ there exists a unique family (f_t) with $f_0 = \varphi$ such that for every $z \in \Delta$ the function $t \mapsto f_t(z)$ is absolutely continuous and satisfies 3.1 *almost eveywhere. Moreover, if φ is injective and such that $\varphi(z) = z + ..$ at ∞ then (f_t) is a Lôwner chain.*

Proof : Suppose first that (f_t) is a Löwner chain : We can then write $f_t(z) = f_s(h_{s,t}(z))$ and thus

$$\frac{f_t(z) - f_s(z)}{t - s} = \frac{f_s(h_{s,t}(z)) - f_s(z)}{h_{s,t}(z) - z}\frac{h_{s,t}(z) - z}{h_{s,t}(z) + z}\frac{e^{t-s} + 1}{e^{t-s} - 1}\frac{e^{t-s} - 1}{t - s}\frac{h_{s,t}(z) + z}{e^{t-s} + 1}.$$

Lemma 3.1.3. *There exists a subset E of $\mathbb{R}_+$ of full measure such that if $s \in E$ then $t \mapsto f_t(z)$ is differentiable at s for every $z \in \Delta$.*

Proof : Let $z \in \Delta$ and (z_k) a sequence of two by two distinct points of Δ converging to z. By the above proposition, for each k, $t \mapsto f_t(z_k)$ is AC and thus there exists a set E_k of full measure such that $t \mapsto f_t(z_k)$ is differentiable on E_k. By Vitali theorem the lemma holds for $E = \cap E_k$.
Take $s \in E$: since $\mathcal{P}(\Delta)$ is a normal family we may choose a sequence (t_n) converging to s such that $p_{s,t_n} \to p_s \in \mathcal{P}(\Delta)$ where

$$p_{s,t}(z) = \frac{h_{s,t}(z) - z}{h_{s,t}(z) + z}\frac{e^{t-s} + 1}{e^{t-s} - 1}.$$

Letting $n \to \infty$ we then obtain (3.1).

We come to the converse. Before starting the proof let us first notice that if (f_t) is a Löwner chain then the function $h_{s,t}(z) = f_s^{-1} \circ f_t(z)$, as a function of s, is a solution of the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}s} = -wp_s(w),\ w(t) = z. \tag{3.2}$$

We reverse the point of view and consider the differential equation (3.2). Since

$$\frac{\mathrm{d}|w|^2}{\mathrm{d}s} = -2|w|^2 \Re p_s(w)$$

the modulus of a solution is decreasing. It follows that the equation (3.2) has a solution $s \mapsto w(s;t,z)$ defined on $[0,t]$. By Cauchy-Lipschitz theorem, this function is injective in z. Moreover

$$\frac{\partial}{\partial s}(f_s(w(s;t,z))) = f_s'(w)\frac{\partial w}{\partial s}(s;t,z) + \frac{\partial f}{\partial s}(w(s;t,z)) = 0.$$

Since $w(t;t,z) = z$, it implies that $\forall s \leq t$, $f_s(w(s;t,z)) = f_t(z)$. Taking $s = 0$ we get $f_t = \varphi(w(0;t,z))$, and the proof of the existence and uniqueness is done. The rest is obvious.

3.1.3 Miscellanea about Löwner equation

Another point of view on Löwner equation

Assume in this section that $f_0 = id$. The computation that has lead to the ODE satisfied by $h_{s,t}$ works as well for $g_t = f_t^{-1}$, and, if $z \in \Delta$, $t \mapsto g_t(z)$ is the solution of the Cauchy problem

$$\dot{g}_t(z) = -g_t(z)p_t(g_t(z)),\ g_o(z) = z.$$

For $z \in \Delta$ let $T(z)$ denote the lifetime of this solution. The next theorem is almost a tautology :

Theorem 3.1.2. *For $t \geq 0$, $K_t \cap \Delta = \{z \in \Delta;\ T(z) \leq t\}$.*

Herglotz representation theorem

Herglotz representation theorem says that the functions $p \in \mathcal{P}(\Delta)$ are precisely the functions

$$p(z) = \int_{-\pi}^{\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} d\mu(\theta)$$

and the correspondence $p \leftrightarrow \mu$ is a bijection. A Löwner process can thus as well be defined by a measurable family (μ_t) of probability measures.
The two extreme choices for these measures are Dirac masses on one side and measures AC wrt Lebesgue measure on the other. Let us briefly study both cases :

First example

We first study the case where $\mu_t = \delta_1$, the Dirac mass at point 1. We claim that this Löwner process corresponds to the growth of the external ray starting at 1. To do this we first compute a conformal mapping from $\mathbb{H}\backslash[0, i\sqrt{t}]$ ($t \in]0,1[$) onto $\mathbb{H}$ and fixing the point i and we find

$$g_t(z) = \sqrt{\frac{z^2+t}{1-t}}.$$

We then conjugate g_t by φ where

$$\varphi(z) = -i\frac{1-z}{1+z}$$

maps Δ on $\mathbb{H}$ to obtain a map G_t from $\Delta\backslash[1,(1+\sqrt{t})/(1-\sqrt{t})]$ onto Δ and the computation gives

$$\dot{G}_t = -\frac{1}{\sqrt{1-t}}G_t\frac{G_t+1}{G_t-1}.$$

A time-change allows to eliminate the constant in front and it becomes clear that the process is then driven by $\mu_t = \delta_1$. Of course the growth along the external ray of argument θ corresponds similarly to the Dirac mass at $e^{i\theta}$.
Let us examine now the process driven by

$$\mu_t = \delta_{e^{i\theta}},\ t \in [0,a], \text{ and } = \delta_{e^{i\theta'}},\ t \in]a,b].$$

Before time a we have already seen that it grows along the external ray θ. In order to understand the process at later times we consider

$$\varphi_t = g_a \circ f_t$$

and its reciprocal

$$\psi_t = g_t \circ f_a.$$

Then

$$\dot{\psi}_t = \dot{g}_t \circ f_a = -\psi_t\frac{\psi_t + e^{i\theta'}}{\psi_t - e^{i\theta'}}$$

for t between a and b, which represents growth along the external ray θ'. Now, because $f_t = f_a \circ \varphi_t$, the growth after time a is the growth along the θ'-external ray of Ω_a.
This can obviously generalized to all step functions, and uniform limits on every compact of step functions are precisely regulated functions, that is the functions having a limit from the left and from the right at every point of $\mathbb{R}_+$. Uniform limits of driving functions are linked to Löwner processes by the following proposition, whose proof is left to the reader :

Proposition 3.1.2. *Let (λ_n) be a sequence of functions defined on $\mathbb{R}_+$ and converging uniformly on compact subsets towards a function λ. Let $f_n(z,t)$ the process driven by λ_n and $f(z,t)$ the one driven by λ. Then $f_n \to f$ uniformly on compact subsets of $\Delta \times \mathbb{R}_+$.*

The statement above involves the important notion of process *driven* by a function :

Definition 3.1.3. *The Löwner process (f_t) is said to be driven by the function $\lambda : \mathbb{R}_+ \to \partial\Delta$ if the familly (p_t) is given by*

$$p_t(z) = \frac{(z+\lambda(t))}{(z-\lambda(t))}.$$

Equivalently the measures μ_t associated to the process are the Dirac masses at $\lambda(t)$.

Growth by equipotentials

A trivial example which is in some sense dual to the growth of external rays is the growth by equipotentials : let f be the Riemann map of a CCF domain and

$$f_t(z) = f(e^t z).$$

This is the Löwner process associated to K_t being the closure of the interior of the Jordan curve $f(e^{t+i\theta})$, $\theta \in [0, 2\pi[$. An obvious computation shows that

$$\dot{f}_t(z) = z f_t'(z),$$

and thus that for this process

$$p_t(z) \equiv z,$$

which corresponds to

$$\mu_t \equiv \frac{|dz|}{2\pi},\ t \geq 0.$$

So in this case μ_t is absolutely continuous. We will see later that Hele-Shaw flows are modelled by a similar but more complicated Löwner process.

3.1.4 Löwner processes driven by regulated functions

As already announced, our aim in this paragraph is to characterize geometrically Löwner processes for which

$$\forall t \geq 0,\ \mu_t = \delta_{\lambda(t)}$$

where $\lambda : \mathbb{R}_+ \to \partial\Delta$ is a regulated function, ie a function having left and right limits at every point. For this study we will have to visit once more Caratheodory theory, and more precisely his notion of prime-end.

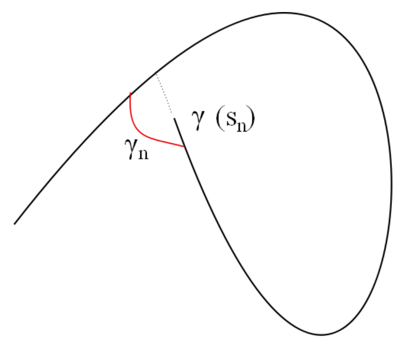


FIG. 3.1 – Left limit

Definition 3.1.4. *Let Ω be a $CCF-$domain : a null-chain of Ω is a sequence of cross-cuts (C_n) such that*
(i) $\mathrm{int}(C_{n+1}) \subset \mathrm{int}(C_n)$,
(ii) $\overline{C}_n \cap \overline{C}_{n+1} = \emptyset$, $n \geq 0$,
(iii) $\mathrm{diam}(C_n) \to 0$ *as* $n \to \infty$.

Strictly speaking, we will only need this notion of null-chain. Nevertheless, for better understanding, let us introduce the notion of prime-end.
A prime-end is an equivalence class of null-chains, the equivalence relation being defined as $(C_n) \sim (C'_n)$ if and only if

$$\forall k > 0, \, \exists n > 0 \, ; \, \mathrm{int}(C_n) \subset \mathrm{int}(C'_k), \, \mathrm{int}(C'_n) \subset \mathrm{int}(C_k).$$

Caratheodory has proven that $\Omega \cup$ the set of prime-ends is a natural compactification of Ω for which the Riemann map extends to an homeomorphism of $\overline{\Delta}$ onto this compactification.
We now have the necessary ingredients for our study. Let (f_t) be a Löwner chain : we denote as usual $\Omega_t = f_t(\Delta)$, $K_t = \overline{\mathbb{C}} \backslash \Omega_t$.

Definition 3.1.5. *We say that (K_t) has a left limit at $t_o > 0$ if for every sequence $s_n \nearrow t_o$ one can find a null-chain (γ_n) of Ω_{t_o} such that for $n \geq 0$, γ_n separates $K_{t_o} \backslash K_{s_n}$ from ∞ in Ω_{s_n}.*

Definition 3.1.6. *We say that (K_t) has a right limit at $t_o \geq 0$ if for any sequence $t_n \searrow t_o$ there exists a null-chain (γ_n) of Ω_{t_o} with $\gamma_n \subset \Omega_{t_n}$ separating $K_{t_n} \backslash K_{t_o}$ from ∞ in Ω_{t_o}.*

The rest of the paragraph is devoted to the proof of the following theorem :

Theorem 3.1.3. *The Löwner process (f_t) is driven by a regulated function if and only if (K_t) has a left and right limit at every point.*

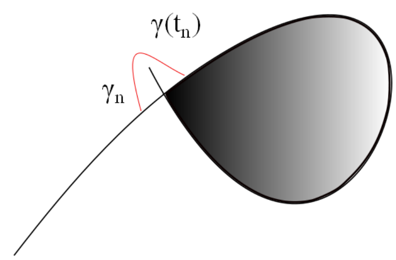


FIG. 3.2 – Right limit

Proof :
1) Assume first that the Löwner process is driven by the regulated function λ :

Proposition 3.1.3. *If λ is right-continuous at t_o then*

$$\operatorname{diam}\left(g_{t_o}(K_{t_o+\delta}\backslash K_{t_o})\right)$$

tends to 0 *as* $\delta \searrow 0$.

Proof : Replacing $t \mapsto g_t$ by $\delta \mapsto h^{-1}_{t_o,t_o+\delta}$ we may assume that $t_o = 0$ (and thus $g_{t_o} = id$) and that the driving function is $\delta \mapsto \lambda(t_o + \delta)$. Define

$$M_t = \max\left(2\sqrt{t}, \sup_{0\le s\le t} |\lambda(s) - \lambda(0)|\right).$$

Assuming t is small enough we can consider $z \in \Delta$ such that $|z| \le 3$ and $|z - \lambda(0)| \ge 10M_t$. We recall that T_z is the lifetime of $s \mapsto g_s(z)$ and define

$$\sigma = \min(T_z, \inf\{s > 0;\ |g_s(z) - z| = M_t\}).$$

We suppose that $t > \sigma$ and consider $s \le \sigma$. Then

$$|\dot{g}_s(z)| = |g_s(z)| \left|\frac{g_s(z) + \lambda(s)}{g_s(z) - \lambda(s)}\right|$$

$$\le 3\frac{4}{|z - \lambda(0)| - 2M_t} \le \frac{3}{2M_t}.$$

We consider then two cases :
First case : $\sigma = T_z$: then $\lambda(T_z) \in D(z, M_t) \cap \partial\Delta$ and $|\lambda(T_z) - \lambda(0)| \ge 9M_t$ in contradiction with the fact that $|\lambda(s) - \lambda(0)| < M_t$ if $s \le t$.
Second case : $\sigma < T_z$: then $|g_\sigma(z) - z| = M_t$, but on the other hand, by the above computation, $|g_\sigma(z) - z| \le \frac{3\sigma}{2M_t}$ and this two facts imply that $M_t^2 \le \frac{3t}{2}$, contradicting the fact that, by definition, $M_t \ge 2\sqrt{t}$.
End of the proof : We have proven, by contradiction, that $\sigma \ge t$ and thus that $T_z \ge t$. This implies that $K_{t_o+\delta}\backslash K_{t_o} \subset D(\lambda(0), 10M_t)$ and the proposition follows.

Proposition 3.1.4. *If λ is left-continuous at $t_o > 0$ then*

$$\mathrm{diam}(g_{t_o}(K_{t_o} \backslash K_{t_o - \delta})$$

)tendsto0as$\delta \searrow 0$.

This statement needs some explanation, since g_{t_o} need not be defined on $K_{t_o} \backslash K_{t_o-\delta}$.
To this end define

$$\varphi_s = h_{t_o - s, t_o} = g_{t_o - s} \circ f_{t_o}.$$

Then φ_s solves

$$\dot{\varphi}_s = \varphi_s \frac{\varphi_s + \lambda(t_0 - s)}{\varphi_s - \lambda(t_0 - s)}$$

with $\varphi_0 = id$. Define as usual T_z as being the life-time of the maximal solution of this equation, this lifetime being equal to 0 if $z = \lambda(t_o)$. We then define $g_{t_o}(K_{t_o} \backslash K_{t_o-\delta})$ in the proposition as being the set of points $z \in \partial\Delta$ such that $T_z \leq \delta$.
We return to the proof of the second proposition and define as before

$$M_t = \max\left(2\sqrt{t}, \sup_{0 \leq s \leq t} |\lambda(t_o) - \lambda(t_o - s)|\right).$$

Let $z \in \partial\Delta$ with $|z - \lambda(t_o)| > 10 M_t$. Put

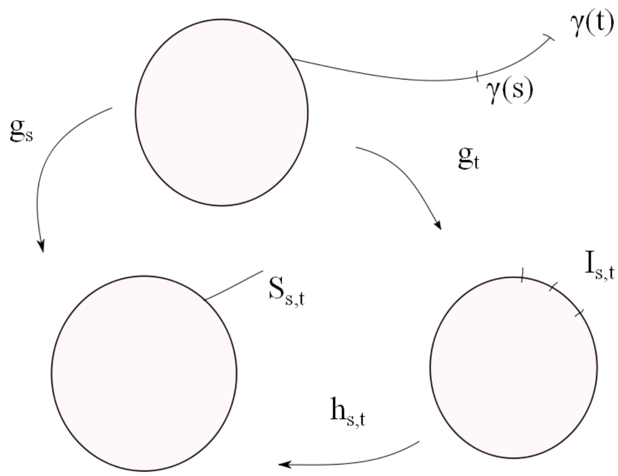
$$\sigma = \min(T_z, \inf\{s > 0;\, |\varphi(z) - z| = M_t\}).$$

Reasoning exactly as in the first proposition, one sees that $\sigma \leq T_z$, and this implies the proposition.
We may now finish the first part of the proof. Consider first a sequence $t_n \searrow t_o$ and put $S_{t_o, t_n} = g_{t_o}(K_{t_n} \backslash K_{t_o})$: by the first proposition this set has small diameter and we can enclose it in the interior of a circular cross-cut of Δ with small radius. By the discussion in Caratheodory's theorem we may assume as well that the length of the image of the cross-cut by f_{t_o} is also small, a fact which proves that (K_t) is right-continuous.
Suppose now that $s_n \nearrow t_o$ and let $I_n = g_{t_o}(K_{t_o} \backslash K_{s_n})$ defined in the second proposition. Notice that by this definition this is an interval and the proposition says that the length of this interval is small. Let ζ_{I_n} be the center of I_n and

$$z_{I_n} = (1 + |I_n|)\zeta_{I_n}.$$

By general theory of univalent functions (Pommerenke, p.350) there exists a cross-cut passing through z_{I_n} in Δ, intercepting an arc containing I_n such that its image by f_{t_o} has length

$$\leq C \mathrm{dist}(f_{t_o}(z_{I_n}), K_{t_o}) \to 0$$



as $n \to \infty$, and the first part of the proof is complete.
2) Assume now that (K_t) has a right and left limit at every point : we want to show that the process is driven by a regulated function.
The first task is to guess what the function λ should be.
First fix $s \geq 0$ and let $t \searrow s$: the sets $g_s(K_t \backslash K_s) = S_{s,t}$ are decreasing and, by the definition of right continuity plus Beurling theorem, one can see that their diameter tends to 0. This allows us to define

$$\{\mu(s)\} = \bigcap_{t>s} \overline{S}_{s,t}.$$

Fix now $t > 0$ and let $s \nearrow t$: Let $I_{s,t} = g_t(K_t \backslash K_s)$ the interval defined in the second proposition above. These intervals are decreasing s $s \nearrow t$ and Beurling theorem combined with the definition of left-continuity show that their lengths must converge to 0. We can thus define

$$\{\lambda(t)\} = \bigcap_{s<t} \overline{I}_{s,t}.$$

Lemma 3.1.4. *We have, for every $t > 0$:*

$$\lim_{s \nearrow t} \mu(s) = \lambda(t)$$

and conversely, for every $s \geq 0$,

$$\lim_{t \searrow s} \lambda(t) = \mu(s).$$

Proof : By Schwarz reflection principle $h_{s,t}$ extends to a function holomorphic outside $\overline{I}_{s,t}$ and $h_{s,t}$ converges to Id as $s \nearrow t$ uniformly on compact subsets of $\overline{\mathbb{C}} \backslash \{\lambda(t)\}$. Let C_ε be the circle of center $\lambda(t)$ and radius ε. If s is close to t then

$\overline{I}_{s,t} \subset D_\varepsilon$. Let also consider a point z outside this circle. Because $h_{s,t}$ is close to the identity z also lies outside $h_{s,t}(C_\varepsilon)$ if s is close to t.
By the argument principle $h_{s,t}$ must take the value z outside C_ε. But $S_{s,t}$ is not attained by points outside C_ε. This implies that $S_{s,t} \subset D_\varepsilon$ for s close enough to t. This implies that μ has a right limit at every point which is equal to λ and a similar reasonning for $h_{s,t}^{-1}$ implies that λ has a left limit at every point which is equal to μ.

Lemma 3.1.5. *The function λ is left-continuous, while μ is right-continuous.*

Proof : (For λ, the proof for μ is similar)
Suppose there exists $s_n \nearrow t$ such that $\lambda(s_n) \to \ell \neq \lambda(t)$. Then, since $\lambda(s_n) = \lim_{k\to\infty} \mu(s_n - 1/k)$ we can find $s'_n \to t$ such that $\mu(s'_n) \to \ell \neq \lambda(t)$, a contradiction.
It follows that the two functions μ, λ are respectively the right and left-continuous regularizations of a regulated function that we may choose to be λ since the set of discontinuities is countable and thus of measure 0.

To derive the Löwner equation we make use of the following formula which is a variant of Cauchy's

$$\log\left(\frac{h_{s,t}(z)}{z}\right) = \frac{1}{2\pi}\int_{\delta_{s,t}} \frac{z+e^{i\theta}}{z-e^{i\theta}} \log|h_{s,t}(e^{i\theta})| d\theta \tag{3.3}$$

If we let $z \to \infty$ in this equality we get

$$t - s = \frac{1}{2\pi}\int_{\delta_{s,t}} \log|h_{s,t}(e^{i\theta})| d\theta$$

and we put $z = g_t(\omega)$ to obtain

$$\log\left(\frac{g_t(\omega)}{g_s(\omega)}\right) = \frac{1}{2\pi}\int_{\delta_{s,t}} \frac{g_t(\omega)+e^{i\theta}}{g_t(\omega)-e^{i\theta}} \log|h_{s,t}(e^{i\theta})| d\theta.$$

Putting everything together we get

$$\frac{\partial}{\partial t}(\log(g_t(\omega))) = \frac{g_t(\omega)+\lambda(t)}{g_t(\omega)-\lambda(t)}$$

from which the equation for f_t follows as in the general case.

3.1.5 Processes generated by a curve

The historical example of Löwner leads to a process driven by a continuous function : this is the process associated with $K_t = \overline{\mathbb{D}} \cup \gamma([0,t])$ where $\gamma : [0,\infty[\to \overline{\Delta}$ is injective with $\gamma(0) = 1$ and $\gamma(t) \in \Delta$, $t > 0$.

This family clearly satisfies the hypothesis of 3.1.3 : the process is thus driven by a regulated function. Moreover the fact that $\gamma(t)$ is not a cut point of K_t implies that $\lambda = \mu$ and thus that the regulated driving function is continuous.
We wish to address the converse : if the Löwner process is driven by a continuous function, is K_t of the preceeding form ? The answer is obvioulsy no as the example described in the picture (3.3) shows. The right guess including this example are

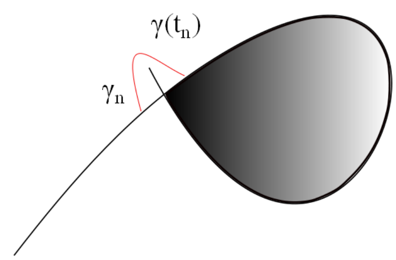


FIG. 3.3 – Non continuous driving function

the processes defined as

Definition 3.1.7. *Let $\gamma : [0, \infty[\to \overline{\Delta}$ be a continuous curve such that $|\gamma(0)| = 1$. We say that the process (Ω_t) is generated by γ if for $t \geq 0$ we have that Ω_t is the unbounded component of the complement of $\overline{\mathbb{D}} \cup \gamma([0, t])$.*

Such processes satisfy the hypothesis of 3.1.3 and thus are driven by a regulated function.
But the converse is again false, even if this is more subtle : the following example is a process driven by a continuous function but not generated by a curve :
Consider the closed disk D of center 3 and radius 1 which is included in Δ. We start a simple curve $\gamma_1 : [0, 1[\to \overline{\Delta}$ at point 1 which spirals towards D and then consider another curve $\gamma_2 :]1, +\infty[\to \delta$ spiraling out D towards ∞ without crossing γ_1. We then define $K_t = \overline{\mathbb{D}} \cup \gamma_1([0, t])$ for $0 \leq t < 1$, $K_1 = \overline{\mathbb{D}} \cup \gamma_1([0, 1[) \cup D$ and, for $t > 1$, $K_t = K_1 \cup \gamma_2(]1, t])$. The compact set K_t is not locally connected for $t \geq 1$ and this fact prevents the process to be generated by a curve. If $s_n \nearrow 1$

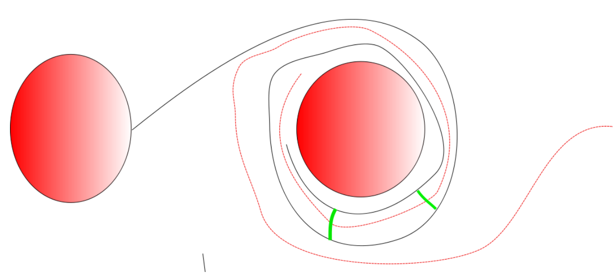


FIG. 3.4 – Driven process not generated by a curve

the depicted cross-cuts θ_n separate $K_1\backslash K_{s_n}$ from ∞ in Ω_{s_n} and if $t_n \searrow 1$ the depicted θ_n' separate $K_{t_n}\backslash K_1$ from ∞ in Ω_1.
It follows from 3.1.3 that the process is driven by a regular function : moreover since from the left and the right at 1 the same null-chain works we must have $\lambda(1) = \mu(1)$, implying that the driving function is actually continuous.

Theorem 3.1.4. *Let (f_t) be a Löwner chain driven by a regulated function λ. This process is generated by a regulated curve if and only if the sets K_t are uniformly locally connected on every compact subsets of $\mathbb{R}_+$.*

Proof : The condition is clearly necessary. Suppose then that the sets (K_t) are uniformly locally connected on compact sets. By [P] p.283 we can deduce that $(t, z) \mapsto f_t(z)$ is actually continuous on $[0, \infty[\times\overline{\Delta}$. If λ stands for the driving function one can then define

$$\gamma(t) = f_t(\lambda(t)),$$

which is a regulated curve : let us show that the process is generated by γ. It is obvious that $\gamma([0, t] \subset K_t)$. It follows that Ω_t is included in the unbounded component of $\overline{\mathbb{C}}\backslash(\overline{\mathbb{D}} \cup \gamma([0, t]))$ that we call U. Suppose that there is a point $z \in K_t \cap U$: let $\delta : [0, \infty[\to U$ be a path joining z to ∞ in U and let $t_o = \max\{s \geq 0; \delta(s) \in K_t\}$ and $z' = \delta(t_o)$. Then z' is accessible in Ω_t : let $s = T_{z'} \leq t$. Since $\Omega_t \subset \Omega_s$, z' is as well accessible in Ω_s.

Lemma 3.1.6. *We have*

$$g_s(z') = \lim_{s' \nearrow s} g_{s'}(z') = \lambda(s) = \lim_{\zeta \to z'} g_s(\zeta),$$

this last limit being taken along the curve δ.

Proof : Put $\xi(u) = \delta(t_o + u)$ so that $z' = \xi(0)$. By theorem 2.7.1 we know that $g_s(\xi(u))$ has a limit as $u \searrow 0$, say ℓ. We need to prove that $\ell = \lambda(s)$. For this end we write

$$|\ell - \lambda(s)| \leq |\ell - g_s(\xi(u))| + |g_s(\xi(u)) - g_{s'}(\xi(u))| + |g_{s'}(\xi(u)) - g_{s'}(\xi(0))| + |g_{s'}(\xi(0)) - \lambda(s)|$$

$$= (I) + (II) + (III) + (IV),$$

where $s' < s$. By definition of ℓ we can make (I) small, as well as (III) by Beurling's estimate, and this uniformly on $s' < s$. Such an u being fixed one can then make (II) and (IV) as small as we wish, finishing the proof of the lemma.
In order to complete the proof of the theorem, we invoke 2.7.1 once more : since $\gamma(s)$ is also accessible and $g_s(\gamma(s)) = \lambda(s)$ we must have $z' = \gamma(s)$, a contradiction.

3.1.6 Whole plane Löwner chains

We develop a variant of the Löwner process called whole-plane Löwner process, because it is better suited for the next paragraph.
In this model we consider a growing family $(\Omega_t)_{t\geq 0}$ of simply-connected domains containing 0 and continuous in the Caratheodory sense. We assume that, at $+\infty$, Ω_t converges to $\mathbb{C}$. We also consider the Riemann map $f_t : \mathbb{D} \to \Omega_t$ satisfying $f_t(0) = 0$, $f_t'(0) > 0$ and assume that $f_0'(0) = 1$. Changing time if necessary we may thus assume that

$$f_t(z) = e^t z + ...$$

Notice that in this model we have that $f_s \prec f_t$ if $s \leq t$: define $h_{t,s}$ such that $f_s = f_t \circ h_{t,s}$. We define a Löwner chain as being a family of injective and holomorphic functions in the unit disk satisfying $f_t(z) = e^t z + ..$ and $f_s \prec f_t$ if $s \leq t$. Notice that by Koebe theorem, if (f_t) is a Löwner chain in this sense, then

$$f_s(z) = \lim_{t\to\infty} e^t h_{t,s}(z).$$

Using the same arguments as in the radial case we can derive for this family of functions a Löwner equation

$$\dot{f}_t(z) = z f_t'(z) p_t(z)$$

with (p_t) being a measurable family of functions in $\mathcal{P}(\mathbb{D})$, the set of functions that are holomorphic in the disk with value 1 at 0 and positive real part.
As for the radial case we can prove a converse of this statement :

Theorem 3.1.5. *Given a measurable family (p_t) of functions in $\mathbb{D}$ there exists a unique Löwner chain (f_t) such that*

$$\dot{f}_t(z) = z f_t'(z) p_t(z)$$

for all $z \in \mathbb{D}$, almosts everywhere in $\mathbb{R}_+$.

Proof : We only outline it since it follows pretty much the same lines as the radial case. Define for $t \geq s$, $h_{t,s}(z)$ as being the value at point t of the solution of the differential equation

$$w'(t) = -w(t) p_t(w(t))$$

with the initial condition $w(s) = z$. This solution is defined on $[s, +\infty[$ because $|w|^2$ is decreasing. By the uniqueness of solutions we see that $z \mapsto e^{t-s} h_{t,s}(z)$ belongs to the class S. Another consequence of this uniqueness is the semi-group property

$$\forall s \leq \tau \leq t,\ h_{t,s} = h_{t,\tau} \circ h_{\tau,s}.$$

From these last two properties and the diagonal process there exists a sequence $t_n \to \infty$ such that the sequence of functions

$$z \mapsto e^{t_n} h_{t_n,s}(z)$$

converges uniformly on compact subsets of $\mathbb{D}$ towards a function that we call f_s and (f_s) is easily seen to be the unique solution of the equation.

3.1.7 Löwner Proof of Bieberbach conjecture for $n = 3$

In this paragraph we develop the result for which Löwner has invented (or discovered) his equation. Let us recall the Bieberbach conjecture : if $f(z) = z + a_2 z^2 + a_3 z^3 + ..$ belongs to the class S then $\forall n \geq 2$, $|a_n| \leq n$. The case $n = 2$ is of course covered by Bieberbach theorem. Considering $f(rz)/r$ it is clear that we may assume that $f(\mathbb{D})$ is a smooth Jordan domain Ω containing 0. Let $\gamma : [0,1] \to \mathbb{C}$ be a parametrization of the boundary of Ω. We continue γ on $[1, +\infty[$ by a simple curve joining $\gamma(1)$ to ∞ inside $\mathbb{C}\backslash\overline{\Omega}$ for $t > 1$.
We consider $\Omega_t = \Omega$ if $t = 0$ and $\Omega_t = \mathbb{C}\backslash\gamma([t, \infty[)$ if $t > 0$. This is a Caratheodory continuous increasing family of domains (where the role of ∞ in the radial case is played here by 0. Let

$$f_t(z) = e^t(z + a_2(t)z^2 + a_3(t)z^3 + ...)$$

be the Riemann map of Ω_t. Using the preceeding paragraph there exists a family (p_t) of functions with positive real part such that 3.1 holds for f_t. Moreover the proof given for radial processes goes true in this new setting : the process is driven by a function. In other words there exists a regulated function $\lambda : [0, \infty[\to \partial\mathbb{D}$ such that

$$p_t(z) = \frac{\lambda(t) + z}{\lambda(t) - z}.$$

The strategy is then to develop at 0 both sides of the Löwner equation and to identify the coefficients. We obtain

$$\dot{a_2} - a_2 = 2\overline{\lambda},$$

$$\dot{a_3} - 2a_3 = 4a_2\overline{\lambda} + 2\overline{\lambda}^2.$$

The first differential equation is easily solved, giving

$$a_2(t) = -2e^t \int_t^\infty \overline{\lambda}(s)e^{-s} ds$$

and a new proof of the $n = 2$ case. Once a_2 is known one can solve the second equation, leading to

$$a_3(t) = -4e^{2t} \int_t^\infty e^{-2s} a_2(s)\overline{\lambda}(s) ds - 2e^{2t} \int_t^\infty e^{-2s}\overline{\lambda}^2(s) ds.$$

We simplify this expression by noticing that the first integral is of the form $\frac{1}{2}\int_t^\infty uu'$ where $u(s) = e^{-s}a_2(s)$. The formula for a_3 then simplifies to

$$a_3(t) = 4e^{2t}(\int_t^\infty \overline{\lambda}(s)e^{-s}ds)^2 - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds.$$

From this last expression it is not hard to derive the estimate $|a_3| \leq 3$. The interested reader can look at a proof in [Conformal Invariants]. We prefer to anticipate a little the next chapters and look at estimates for coefficients for the whole-plane SLE_κ process.
SLE_κ is the Löwner process driven by the function

$$\lambda(t) = e^{i\sqrt{\kappa}B_t}$$

where B_t is a standard one dimensional brownian motion. For such a process we will call a_n the Taylor coefficients of $f = f_0$.

Theorem 3.1.6. *For SLE_κ we have*

$$E(|a_2|^2) = \frac{8}{2+\kappa}.$$

We recall the expressions for a_2 and a_3 :

$$a_2(t) = -2e^t\int_t^\infty \overline{\lambda}(s)e^{-s}ds$$

,

$$a_3(t) = 4e^{2t}(\int_t^\infty \overline{\lambda}(s)e^{-s}ds)^2 - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds.$$

We can thus write

$$|a_2|^2 = 4\int_0^\infty\int_0^\infty e^{-(s+s')}e^{-i\sqrt{\kappa}(B_s-B_{s'})}dsds'.$$

Using now that $B_s - B_{s'}$ follows a normal law with variance $|s-s'|$ we get

$$E(|a_2|^2) = 4\int_0^\infty\int_0^\infty e^{-(s+s')}e^{-\frac{\kappa|s-s'|}{2}}dsds' = 8\int_0^\infty\int_s^\infty e^{-(s+s')}e^{-\frac{\kappa(s-s')}{2}}dsds',$$

and an easy computation gives the result. We now pass to computations involving a_3. In order to avoid repetitions of computations we will compute

$$E(|a_3 - \mu a_2^2|^2)$$

where μ is a real constant. By the above computations,

$$a_3(t) - \mu a_2(t)^2 = 4(1-\mu)e^{2t}(\int_t^\infty \overline{\lambda}(s)e^{-s}ds)^2 - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds.$$

We may then write

$$e^{-4t}|a_3 - \mu a_2^2|^2 = 16(1-\mu)^2 I_1 - 16(1-\mu)\Re I_2 + 4I_3$$

where :

$$I_1 = \int_t^\infty \int_t^\infty \int_t^\infty \int_t^\infty e^{-(s_1+s_2+s_3+s_4)} \overline{\lambda}(s_1)\lambda(s_2)\overline{\lambda}(s_3)\lambda(s_4) ds_1 ds_2 ds_3 ds_4,$$

$$I_2 = \int_t^\infty \int_t^\infty \int_t^\infty e^{-(s_1+s_2+2s_3)} \overline{\lambda}(s_1)\overline{\lambda}(s_2)\lambda(s_3)^2 ds_1 ds_2 ds_3,$$

$$I_3 = \int_t^\infty \int_t^\infty e^{-2(s_1+s_2)} \overline{\lambda}(s_1)^2 \lambda(s_2)^2 ds_1 ds_2.$$

From now on we put $t = 0$ in the above formulas.The computation of I_3 follows the same lines as the one in the theorem 3.1 :

$$I_3 = \int_0^\infty \int_0^\infty e^{-2(s_1+s_2)} e^{-2i\sqrt{\kappa}(B_{s_1} - B_{s_2})} ds_1 ds_2,$$

so that

$$E(I_3) = \int_0^\infty \int_0^\infty e^{-2(s_1+s_2)} e^{-2\kappa|s_1-s_2|} ds_1 ds_2 = 2\int_0^\infty \int_{s_1}^\infty e^{-2(s_1+s_2)} e^{-2\kappa(s_2-s_1)} ds_1 ds_2,$$

from where we easily get

$$E(I_3) = \frac{1}{4(1+\kappa)}.$$

To compute $E(I_2)$ we have to use the strong Markov property satisfied by Brownian motion. First, by symmetry, we may write

$$I_2 = 2\int_{s_1=0}^\infty \int_{s_2=s_1}^\infty \int_{s_3=0}^\infty e^{-(s_1+s_2+2s_3)} e^{i\sqrt{\kappa}(B_{s_3} - B_{s_1})} e^{i\sqrt{\kappa}(B_{s_3} - B_{s_2})} ds_1 ds_2 ds_3$$

and we cut this integral as $I_2 = 2(I_{2,1} + I_{2,2} + I_{2,3})$ where in $I_{2,1}$ (resp. in $I_{2,2}, I_{2,3}$) , s_3 ranges in $[0, s_1]$ (resp. in $[s_1, s_2], [s_2, +\infty[$). For $I_{2,1}$ we write

$$e^{i\sqrt{\kappa}(B_{s_3} - B_{s_1})} e^{i\sqrt{\kappa}(B_{s_3} - B_{s_2})} = e^{2i\sqrt{\kappa}(B_{s_3} - B_{s_1})} e^{i\sqrt{\kappa}(B_{s_1} - B_{s_2})}$$

so that we can use Markov property and deduce that the expectation of this random variable is

$$e^{-2\kappa(s_1-s_3)} e^{-\frac{\kappa}{2}(s_2-s_1)}$$

. From this the value of $E(I_{2,1})$ can be easily computed and we find

$$E(I_{2,1}) = \frac{1}{4(1+\kappa)(2+\kappa)}.$$

For $E(I_{2,2})$ we can directly use Markov property and this leads to

$$E(I_{2,2}) = \frac{1}{(2+\kappa)(6+\kappa)}$$

. For the computation of $E(I_{2,3})$ we write

$$e^{i\sqrt{\kappa}(B_{s_3}-B_{s_1})}e^{i\sqrt{\kappa}(B_{s_3}-B_{s_2})} = e^{2i\sqrt{\kappa}(B_{s_3}-B_{s_2})}e^{i\sqrt{\kappa}(B_{s_2}-B_{s_1})}$$

and the rest, which is similar, leads to $E(I_{2,3}) = \frac{1}{4(1+\kappa)(6+\kappa)}$. Combining these computations we get

$$E(I_2) = \frac{3}{(1+\kappa)(6+\kappa)}.$$

The computation of I_1 follows the same lines. First, by symmetry,

$$I_1 = 4\int_0^\infty \int_{s_1}^\infty \int_0^\infty \int_{s_3}^\infty e^{-(s_1+s_2+s_3+s_4)}e^{i\sqrt{\kappa}(B_{s_3}-B_{s_1})}e^{i\sqrt{\kappa}(B_{s_4}-B_{s_2})}ds_1ds_2ds_3ds_4.$$

We then split this integral into a sum of six pieces according to :
(I) $s_3 < s_4 < s_1 < s_2$,
(II) $s_3 < s_1 < s_4 < s_2$,
(III) $s_3 < s_1 < s_2 < s_4$,
(IV) $s_1 < s_3 < s_4 < s_2$,
(V) $s_1 < s_3 < s_2 < s_4$,
(VI) $s_1 < s_2 < s_3 < s_4$.
Clearly $(I) = (VI)$, $(II) = (V)$, $(III) = (IV)$. Using the same arguments as in the previous computations, skipping the details, we get

$$E((I)) = \frac{1}{(1+\kappa)(2+\kappa)(6+\kappa)},\ E((II)) = \frac{1}{2(2+\kappa)(6+\kappa)},\ E((III)) = \frac{1}{2(2+\kappa)(6+\kappa)}.$$

Altogether we get

$$E(I_1) = \frac{4(3+2\kappa)}{(1+\kappa)(2+\kappa)(6+\kappa)}.$$

We may now state :

Theorem 3.1.7. *If μ is a real coefficient then*

$$E(|a_3 - \mu a_2^2|^2) = \frac{(108 - 288\mu + 192\mu^2) + (88 - 208\mu + 128\mu^2)\kappa + \kappa^2}{(1+\kappa)(2+\kappa)(6+\kappa)}.$$

Let us now develop some corollaries of this last formula :
The first corollary is the analogue of Löwner's estimate, i.e. the value obtained by taking $\mu = 0$.

Theorem 3.1.8.
$$E(|a_3|^2) = \frac{108 + 88\kappa + \kappa^2}{(1+\kappa)(2+\kappa)(6+\kappa)}.$$

The second corollary shows that there is no
Fekete-Szegö counter-example in the SLE family.
We start with $f \in S$ being f_0 for a SLE_κ
process. We associate to it the odd function h as above,
that is $h(z) = z\sqrt{\frac{f(z)}{z}} = z + b_3 z^3 + b_5 z^5 + ...$
while $f(z) = z + a_2 z^2 + a_3 z^3 + ...$
An easy computation gives

$$b_5 = \frac{1}{2}(a_3 - \frac{1}{4}a_2^2).$$

We thus put $\mu = \frac{1}{4}$
in the above theorem and get

$$E(|a_5|^2) = \frac{12 + 44\kappa + \kappa^2}{(1+\kappa)(2+\kappa)(6+\kappa)},$$

a value which is always less or equal to 1 (equal to 1 for $\kappa = 0$).
The last corollary concerns the schwarzian derivative, whose definition is

$$S_f(z) = \frac{f'''(0)}{f'(0)} - \frac{3}{2}(\frac{f''(0)}{f'(0)})^2$$

. We easily obtain $S_f(0) = 6(a_3 - a_2)^2$, thus corresponding to $\mu = 1$. The result is

$$E(|S_f(0)|^2) = \frac{36}{1+\kappa}.$$

A few comments about these results :
1) It is striking that $E(|a_2|^2) = E(|a_3|^2) = 1$ for $\kappa = 6$. Is it a coïncidence or a general fact ?
2) For all values of κ we have $E(|a_5|^2) \leq 1$: there is no Fekete-Szegö counterexample in the SLE-family. Does it remain for higher order terms ? This is not clear since the formulas are complicated and that it is not clear if the values of the expectations are decreasing as a function of κ.
3) It is known that $|S_f(0)|^2 \leq 6$ whenever f is injective. Conversely if $(1 - |z|^2|S_f(z)|^2 \leq 6 \leq 2$ then f is injective ; in our case the value of 2 is reached for $\kappa \geq 8$. What is the interpretation of this fact ?
..

3.2 Chordal Löwner equation

The Löwner processes driven by a regulated function that we have encountered so far were the radial ones , starting at a point on the boundary of Δ and going to ∞ which is an interior point of Δ. The variant of Löwner process we want to define here start at a boundary point and head on to another boundary point. The convenient geometry for this new family of processes is the upper half-plane $\mathbb{H} = \{y > 0\}$, the starting and target points being respectively 0 and ∞.
In the theory of radial Löwner processes an important role is played by logarithmic capacity. An analogue role is played here by another capacity, called half-plane-capacity or hcap. The next section develops the theory of this new capacity.

3.2.1 Half-plane capacity

Definition 3.2.1. *A bounded set $A \subset \overline{\mathbb{H}}$ is called a compact $\mathbb{H}$-hull if $\mathbb{H} \backslash A$ is simply connected.*

We will denote by $\mathcal{Q}$ the set of compact hulls : if $A \in \mathcal{Q}$ we define $A^* = A \cup \mathcal{C}(A) \cup [\inf(A), \sup(A)]$ where $\mathcal{C}(A) = \{\overline{z},\ z \in A\}$. The set A^* is a $CCF-$set.

Proposition 3.2.1. *If $A \in \mathcal{Q}$ there exists a unique mapping $g_A : \mathbb{H} \backslash A \to \mathbb{H}$ holomorphic and bijective such that*

$$\lim_{z \to \infty} g_A(z) - z = 0.$$

This mapping is said to satisfy the hydrodynamic normalization.
Proof : First of all by 2.7.1 and Riemann mapping theorem there exists an holomorphic bijection $g : \mathbb{H} \backslash A \to \mathbb{H}$ such that

$$\lim_{z \to \infty} g(z) = \infty.$$

This map extends, by Schwarz reflection principle, to a mapping $g^* : \overline{\mathbb{C}} \backslash A^* \to \mathbb{C}$ with $g^*(\infty) = \infty$. It has a Laurent development at ∞

$$g^*(z) = \lambda z + b_o + \frac{b_1}{z} + ...$$

Notice that λ must be real since

$$\overline{g^*(\overline{z})} = g^*(z).$$

It follows that

$$g_A(z) = \frac{g^*(z) - b_o}{\lambda}$$

has the desired property. Assume that φ_1, φ_2 are two mappings satisfying the desired property then $\varphi = \varphi_1 \circ \varphi_2^{-1}$ is an automorphism of $\mathbb{H}$ such that $\varphi(z) - z \to 0$ as $z \to \infty$. This implies that $\varphi = id$.
The mapping g_A admits the Laurent expansion

$$g_A(z) = z + \frac{b_1}{z} + ...$$

Definition 3.2.2. *The coefficient b_1 is called the Half-space capacity of A and denoted by* $\mathrm{hcap}(A)$*.*

The terminology is justified by the following proposition :

Proposition 3.2.2. *For every $A \in \mathcal{Q}$ we have* $\mathrm{hcap}(A) \geq 0$*. Moreover, if $A \subset A' \in \mathcal{Q}$ then* $\mathrm{hcap}(A) \leq \mathrm{hcap}(A')$*.*

Proof : The fact that $\mathrm{hcap}(A)$ is real follows by symmetry. To prove that it is nonnegative we consider the function

$$v(z) = \Im(z - g_A(z)).$$

This function is harmonic in $\mathbb{H} \backslash A$ and

$$\lim_{z \to \infty} v(z) = 0.$$

Moreover,

$$\liminf_{z \to \zeta \in \partial(\mathbb{H})} v(z) \geq 0.$$

The result then follows from the maximum principle.
Let us now consider $A \subset A' \in \mathcal{Q}$.
As the picture suggests we have

$$g_{A'} = g_B \circ g_A$$

where $B = g_A(A')$ and thus, at ∞,

$$g_{A'}(z) = z + \frac{\mathrm{hcap}(A')}{z} + .. = g_A(z) + \frac{\mathrm{hcap}(B)}{z} + ..,$$

so that $\mathrm{hcap}(A') = \mathrm{hcap}(A) + \mathrm{hcap}(B) \geq \mathrm{hcap}(A)$ by the first part of the proposition.

Proposition 3.2.3. *If $A \in \mathcal{Q}$ is such that* $\mathrm{hcap}(A) = 0$ *then $A \cap \mathbb{H} = \emptyset$ (equivalently $A \subset \mathbb{R}$).*

Proof : We may assume that A is not a singleton : there then exists a Riemann map from Δ onto $\overline{\mathbb{C}}\backslash A^*$. This map has the development at ∞

$$f(z) = \text{cap}(A^*)[z + b_o + \frac{b_1}{z} + ...]$$

and $|b_1| \leq 1$ by area theorem. If we denote by φ the mapping

$$\varphi(z) = g_{\overline{\mathbb{D}}\cap\overline{\mathbb{H}}} = z + \frac{1}{z},$$

we have that $G_A = \varphi \circ f^{-1}$ maps $\mathbb{H}\backslash A$ onto $\mathbb{H}$. Since

$$f^{-1}(z) = \frac{z}{\text{cap}(A^*)} - b_o - \frac{b_1 \text{cap}(A^*)}{z} + ..,$$

we have

$$G_A(z) = \frac{z}{\text{cap}(A^*)} - b_o - \frac{b_1 \text{cap}(A^*)}{z} + \frac{\text{cap}(A^*)}{z} + ...$$

We deduce from this computation the link between hcap(A) and cap(A^*), namely :

$$\text{hcap}(A) = (1 - b_1)(\text{cap}(A^*))^2. \tag{3.4}$$

We can now finish the proof : since cap$(A^*) > 0$, the fact that hcap$(A) = 0$ implies $b_1 = 1$ and thus by area theorem that

$$f(z) = k[z + b_o + \frac{1}{z}]$$

and that $A \subset \mathbb{R}$.

Even if hcap is a capacity, it does not scale like logarithmic capacity : we have seen that if K is a $CCF-$set and $r > 0$ then

$$\text{cap}(rK) = r\text{cap}(K).$$

An obvious inspection shows that, on the opposite, for $A \in \mathcal{Q}$,

$$\text{hcap}(rA) = r^2 \text{hcap}(A).$$

Notice that, by (3.4),

$$\text{hcap}(A) \geq \frac{|A^*|}{\pi}.$$

The opposite inequality is impossible since sets with zero area may have positive hcap. It can be shown (S.Rohde, personnal communication) that hcap(A) is comparable to the sum of the areas of the Whitney squares that intersect A.

3.2.2 Chordal Löwner processes

In this section we describe the theory of chordal growth processes. Most of the proofs are exactly the same that in the radial case. We will thus omit them and stick to the description, with emphasis on the differences.
As in the radial case we start by considering a growing family K_t of compact sets in $\mathcal{Q}$ with $K_0 = \emptyset$. We denote as in the radial case $\Omega_t = \mathbb{H} \backslash K_t$ and we make two assumptions :
1) We assume that for all $t > 0$ the set $\cup_{s<t} K_s$ is bounded.
2) We assume also as in the radial case that the family $(\Omega_t)_{t \geq 0}$ is continuous wrt Caratheodory convergence. Let for $t \geq 0, g_t = g_{K_t}$ be the map with hydrodynamic normalization defined above and $f_t = g_t^{-1}$ that we define as being the Riemann map of the $\mathbb{H}$-domain Ω_t. Caratheodory convergence theorem asserts that the family $(f_t)_{t \geq 0}$ is continuous for the topology of uniform convergence on compact subsets of $\mathbb{H}$: notice that f_t has a holomorphic extension to a neighborhood of ∞ (because K_t is compact) and therefore the function

$$t \mapsto \mathrm{hcap}(K_t)$$

is continuous. We make the further asumption that this function, which is increasing by the preceeding paragraph, converges to $+\infty$ with t. Changing time if necessary we can assume that

$$\mathrm{hcap}(K_t) = 2t, \, t \geq 0.$$

Reasonning as in the radial case we see that such a process is AC wrt the Lebesgue measure. The growth condition means, at points of differentiability, that

$$\Im \left(\frac{\dot{f}_t}{f_t'} \right) > 0$$

and so there exists a measurable family of probability measures μ_t on the real line such that

$$\dot{f}_t(z) = 2 f_t'(z) \int_{\mathbb{R}} \frac{d\mu_t(x)}{x - z}, \, \text{for} \, a.e. \, t > 0 \, , \forall z \in \mathbb{H}.$$

Moreover the fact that K_t is a $\mathbb{H}-$hull implies that μ_t has compact support and the first assumption above implies that the union of the supports of the μ_s, $s \leq t$ is bounded.
This is the half-plane version of the Löwner partial differential equation. There is also a version of the Löwner differential equation : this is the equation satisfied by g_t which reads

$$\dot{g}_t(z) = 2 \int_{\mathbb{R}} \frac{d\mu_t(x)}{g_t(z) - x}.$$

As in the radial case the set Ω_t is then the set of points $z \in \mathbb{H}$ such that the life-time of the maximal solution with value z at 0 is $> t$.
The important fact about this theory is, as in the radial case, its converse :
So let us consider a family of probability measures $(\mu_t)\, t \geq 0$ such that for every $t \geq 0$ there exists $M_t < +\infty$ such that

$$\forall s \leq t, \, \mathrm{supp}(\mu_t) \subset [-M_t, M_t].$$

For each $z \in \mathbb{H}$ we can consider the Cauchy problem :

$$y'(t) = 2\int_{\mathbb{R}} \frac{d\mu_t(x)}{y(t) - x}, \, y(0) = z.$$

We call $g_t(z)$ the solution of this equation : then g_t is a holomorphic bijection from $\{T_z > t\}$ (T_z is the life-time of the solution) onto $\mathbb{H}$ with hydrodynamic normalization.
This last statement requires some explanation : to make it rigorous one must establish that $\{T_z \leq t\}$ is bounded and justify the development at ∞.

Lemma 3.2.1. *For $t \geq 0$ there exists $R_t > 0$ such that g_t extends to a holomorphic function outside the disk of center 0 and radius R_t. Moreover g_t has the following Laurent expansion at ∞ :*

$$g_t(z) = z + \frac{2t}{z} + \ldots$$

Proof : Consider the real M_t defined above : if $|z| \geq (k+1)M_t$ define T to be the first time that $|g_t(z) - z| = 1$. Let $k > t/M_t$: if $|k| \geq t/M_t$ then, if $|z| > (k+1)M_t$, we must have $T > t$. For the rest of the proof one simply write $g_t(z) = \lambda(t)z + \mu + \nu/z + \ldots$ and identify in the Löwner differential equation.
As in the radial case we will say that the Löwner process is driven by a function if for every $t \geq 0$ we have $\mu_t = 2\delta_{\lambda(t)}$ for a function $\lambda : \, [0, +\infty[\to \mathbb{R}$. The Löwner equations then reads :

$$\dot{f}_t(z) = \frac{2f_t'(z)}{\lambda(t) - z},$$

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)}.$$

The characterization of processes driven by a regulated function in the radial case goes through in the chordal case without change. The same is true for the characterization of processes generated by curves.
We wish to define now SLE_κ processes :

Definition 3.2.3. *For $\kappa \geq 0$ we define the SLE_κ process as the Löwner process defined through the differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}$$

where B_t stands for a standard Brownian motion on $\mathbb{R}$.

We will discuss these processes in chapter 5. In the next one we give a precise definition of Brownian motion and develop Ito calculus which will be a key tool later.

Chapitre 4

Stochastic Processes and Brownian Motion

4.1 Construction of Brownian Motion

We consider a probability space $(\Omega, \mathcal{F}, P)$, a measurable set $E, \mathcal{E}$ and a set T.

Definition 4.1.1. *: A stochastic process indexed by T and with values in E is a family (X_t) of measurable functions $\Omega \to E$.*

The space E is the state space while T represents time. Most of the time $T = \mathbb{N}$ or $\mathbb{Z}$ (discrete case) or $\mathbb{R}_+$, $\mathbb{R}$ (continuous case).
We now proceed to construct the most important stochastic process, i.e. Brownian Motion (BM). To this end we start with the

Proposition 4.1.1. *:Let H be a separable Hilbert space. There exists a probability space $(\Omega, \mathcal{F}, P)$ and a family (X_h), $h \in H$ of real random variables such that*
- *(i) $h \mapsto X_h$ is linear,*
- *(ii) X_h is for $h \in H$ a centered Gaussian variable with*

$$E(X_h)^2 = \|h\|^2.$$

Proof : Consider an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. We know that there exists a probability space and a sequence of independent reduced Gaussian variables. It then suffices to define $X_h = \sum_{n\geq 0} < h, e_n > g_n$.

Definition 4.1.2. *:When $H = L^2(A, \mathcal{A}, \mu)$ then the mapping $h \mapsto X_h$ is a Gaussian measure with intensity μ.*

The reason for this definition is that we can define for $F \in \mathcal{A}$; $\mu(F) < \infty$, $X(F) = X_{1_F}$. Since in a Gaussian space L^2 convergence and almost sure convergence are equivalent it is true that if $\mu(F) < \infty$, $F = \cup F_n$ then

$$X(F) = \sum_{n=0}^{\infty} X(F_n)$$

a.s. It is not true though that for almost all ω, $F \mapsto X(F)(\omega)$ is a measure. Let us also notice that if $F, G \in \mathcal{A}$, $\mu(F)$, $\mu(G) < \infty$, then

$$E(X(F)X(G)) = \mu(F \cap G).$$

Let us start the construction of BM. We put $(A, \mathcal{A}, \mu) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \text{Lebesgue measure})$ and for each $t \geq 0$ we choose an element B_t in the class $X([0,t])$. Let us study the properties of this stochastic process :
1) By the last remark above this process has independent increments, i.e. if $t_0 < t_1 < ... < t_n$ the variables $B_{t_{i+1}} - B_{t_i}$ are independent for $i = 0, ..., n-1$.
2) With the same notations the (vectorial) variable $(B_{t_0}, B_{t_1}, ..., B_{t_n})$ is Gaussian.
3) For each t, $E(B_t^2) = t$.
To have a good definition of Brownian motion we need further the paths $t \mapsto X_t(\omega)$ to be a.s. continuous. But in order for this statement to be meaningful we need that the set of $\omega's$ for which the path is continuous to be measurable, and there is no reason for that. To overcome this difficulty we will use the following notions :

Definition 4.1.3. *: Two processes X, X' (not necessarilly defined on the same probability space but with the same state space) are said to be a version of each other if for every sequence of times $t_1, .., t_n$, the variables $(X_{t_1}, .., X_{t_n})$ and $(X'_{t_1}, .., X'_{t_n})$ have the same law.*

Definition 4.1.4. *: Two processes X, X' defined on the same probability space and with the same state space are said to be a modification of each other if for every t, a.s. $X_t = X'_t$. They are called indistinguishable if a.s. $\forall t, X_t(\omega) = X'_t(\omega)$.*

If two processes are modifications of each other then they are versions of each other. Also, since a continuous function on $\mathbb{R}$ is determined by its values on $\mathbb{Q}$ two processes that are a.s. continuous and that are modifications of each other are indistinguishable.

Theorem 4.1.1. *(Kolmogorov's criterium) A real-valued process for which there exists $\alpha \geq 1$, β, $C > 0$ such that for every t, h*

$$E[|X_{t+h} - X_t|^\alpha] \leq Ch^{1+\beta}$$

has a modification which is almost-surely continuous.

Proof : We put, for $j \in \mathbb{N}$, $K_j = \sup\{|X_t - X_s|,\ t,\ s$ dyadic of order j, $|t-s| = 2^{-j}\}$.
Then $E(K_j^\alpha) \leq \sum_{\text{all possible s,t}} E[|X_t - X_s|^\alpha] \leq 2^j c 2^{-j(1+\beta)} = c2^{-j\beta}$. Let now s, t be two dyadic number in $[0,1]$ such that $|s-t| \in [2^{-m-1}, 2^{-m}]$. Let s_j, t_j be the biggest dyadic numbers of order j which are $\leq s, t$. Then

$$X_s - X_t = \sum_m^\infty (X_{s_{j+1}} - X_{s_j}) + (X_{s_m} - X_{t_m}) + \sum_m^\infty (X_{t_{j+1}} - X_{t_j})$$

from which it follows that

$$|X_t - X_s| \leq 2 \sum_m^\infty K_j.$$

Let us then define

$$M_\gamma = \sup\{\frac{|X_t - X_s|}{|t-s|^\gamma}, s \neq t\,\text{dyadic}\}.$$

Then

$$M_\gamma \leq C \sup_{m \in \mathbb{N}} (2^{m\gamma} \sum_m^\infty K_j) \leq C \sum_0^\infty 2^{j\gamma} K_j.$$

Now

$$\left(E(M_\gamma^\alpha)^{1/\alpha}\right) \leq C \sum 2^{j\gamma} \left(E(K_j^\alpha)^{1/\alpha}\right) \leq C \sum 2^{j(\gamma\alpha - \beta)} < \infty$$

if $\gamma < \beta/\alpha$. It follows that a.s. $t \mapsto X_t(\omega)$ is uniformly continuous on the dyadics and thus has a unique extension $\tilde{X}_t$ continuous on $\mathbb{R}$. By Fatou's lemma $\tilde{X}_t$ is the desired modification. The theorem applies in our situation since $B_{t+h} - B_t$ is Gaussian centered with variance h because then

$$E(B_{t+h} - B_t)^{2p} = C_p h^p.$$

We more precisely get that a.s. $t \mapsto B_t$ is $\gamma-$Hölder $\forall \gamma < 1/2$.

4.2 Canonical processes

If X is a stochastic process then for each ω we may view $t \mapsto X_t(\omega)$ as a map from T in E, i.e. an element of $\mathcal{F}(T, E) = E^T$. thus if $w \in E^T$ we think of $w(t), t \in T$ as the coordinates of w that we denote $Y_t(Y_t(w) = w(t))$. Now we can endow E^T with the product σ-algebra (E^T), i.e. the smallest σ-algebra making all the coordiante mappings Y_t measurable. It can also be described as the σ-algebra generated by the products $\prod A_t$ where $A_t = E$ for all $t \in T$ except a finite number for which $A_t \in \mathcal{E}$. We now return to our process X and define a map from Ω in E^T by

$$\Phi(\omega)(t) = X_t(\omega).$$

This mapping is measurable by definition of (E^T). Let us call P_X the image of P by Φ ; the processes X_t, P and Y_t, P_X are then versions of each other.

Definition 4.2.1. *We call Y the canonical version of X and P_X the law of X.*

If the process X has continuous paths with $T = \mathbb{R}_+$ we can proceed as before on the space $C(\mathbb{R}_+, E)$. Doing this with BM we then get

Theorem 4.2.1. *There exists a unique probability measure W on $C(\mathbb{R}_+, \mathbb{R})$ for which the coordinate process is a Brownian motion. It si called the Wiener measure on the Wiener space $C(\mathbb{R}_+, \mathbb{R})$.*

4.3 Filtrations and stopping times

Definition 4.3.1. *: A filtration on a measurable space $(\Omega, \mathcal{F})$ is an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$. A measurable space endowed with a filtration is called a filtered space.*

Definition 4.3.2. *: A process (X_t) on a filtered space is called adapted to the filtration if $\forall t \geq 0$, X_t is $\mathcal{F}_t$ measurable.*

Any process is adapted to its natural filtration $\mathcal{F}_t^0 = \sigma(X_s, s \leq t)$ which is the smallest filtration to which X is adapted. We define for any filtration

$$\mathcal{F}_t^- = \bigvee_{s<t} \mathcal{F}_s,\ \mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s,\ \mathcal{F}_\infty = \bigvee_{s\geq 0} \mathcal{F}_s.$$

Definition 4.3.3. *A stopping time relative to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a map $T : \Omega \to [0, +\infty]$ such that for every $t \geq 0, \{T \leq t\} \in \mathcal{F}_t$.*

If T is a stopping time we define $\mathcal{F}_T$ as the σ-algebra of sets A such that $A \cap \{T \leq t\} \subset \mathcal{F}_t, t \geq 0$.

Proposition 4.3.1. *: If E is a metric space and if X is the coordinate process on $C(\mathbb{R}_+, E)$ then if $A \subset E$ is closed then*

$$D_A(\omega) = \inf\{t \geq 0; X_t(\omega) \in A\}$$

is a stopping time for its natural filtration.

Proof : $\{D_A \leq t\} = \{\omega; \inf\{d(X_s(\omega), A), s \in \mathbb{Q}, s \leq t\} = 0\}$.

4.4 Martingales

In what follows we always have a probability space $(\Omega, \mathcal{F}, P)$, an interval T of $\mathbb{N}$ or $\mathbb{R}_+$ and a filtration $(\mathcal{F}_t)_{t\in T}$ of sub $\sigma - algebras$ of $\mathcal{F}$.

Definition 4.4.1. *A real-valued process (X_t), $t \in T$ such that $\forall t \in T$, $E(|X_t|) < +\infty$ which is $\mathcal{F}_t-$adapted is called a sub-martingale (resp. a super-martingale, resp. a martingale) if*
$\forall s < t \in T$, $X_s \leq E[X_t|\mathcal{F}_s]$ (resp. $X_s \geq E[X_t|\mathcal{F}_s]$, resp. $X_s = E[X_t|\mathcal{F}_s]$).

The two following propositions are versions, valid for essentially finite martingales, of the very general optional stopping theorem to be stated below.

Proposition 4.4.1. *If (X_n) is a martingale and (H_n) is a positive bouded process such that for $n \geq 1, H_n$ is $\mathcal{F}_{n-1}-$measurable. Then the process*

$$Y_0 = X_0,\ Y_n = Y_{n-1} + H_n(X_n - X_{n-1})$$

is a martingale.

Proof : Obvious.
We denote by $H \cdot X$ the process defined in this proposition. As will become clear later, this is a discrete version of Ito's stochastic integral.

Corollary 4.4.1. *With the same notations, if T is a stopping time, then the stopped process $X^T = X_{T\wedge t}$ is a martingale.*

Proof : it suffices to apply the preceeding proposition to $H_n = 1_{T\geq n}$.
We come now to a first version of the optional stopping theorem :

Theorem 4.4.1. *If $S \leq T$ are two bounded stopping times and (X_n) is a martingale then $X_S = E(X_T|\mathcal{F}_S)$.*

Proof : If $M \in \mathbb{R}$ is such that $S \leq T \leq M$ then, putting $H_n = 1_{T\geq n} - 1_{S\geq n}$, we have

$$(H \cdot X)_n - X_0 = X_T - X_S$$

if $n > M$ and it follows that $E(X_S) = E(X_T)$.If we apply this equality to the stopping times $\tilde{S} = S1_B + M1_{^cB}$, $\tilde{T} = T1_B + M1_{^cB}$ with $B \in \mathcal{F}_S$ we get that

$$E[X_T 1_B] = E[X_S 1_B]$$

i.e. the desired result.

4.4.1 Maximal inequalities

Theorem 4.4.2. *Let X be a (sub-)martingale indexed by $T = \{1, ..., N\}$ then for every $p \geq 1$, $\lambda > 0$,*

$$\lambda P(\{\sup_{t\in T} |X_t| \geq \lambda\}) \leq \int_{\sup_n(|X_n|)\geq\lambda} |X_N|dP).$$

Proof : The process $(|X_n|)$ is a submartingale : λ being fixed we intoduce the stopping time $T = \inf\{n; X_n \geq \lambda\}$ if this set is not empty, and $T = N$ otherwise. By the previous results

$$E(|X_N|) \geq E(|X_T|) = \int_{\sup_n(|X_n|)\geq\lambda} \sup_n(|X_n|)dP + \int_{\sup_n(|X_n|)<\lambda} |X_N|dP$$

$$\geq \lambda P(\sup(|X_n|) > \lambda) + \int_{\sup_n(|X_n|)<\lambda} |X_N|dP.$$

Substracting $\int_{\{} \sup_n(|X_n|) < \lambda\}|X_N|dP$ from the first and last term we get what we want.

Corollary 4.4.2. *With the hypothesises of the preceeding theorem, denoting $X^* = \sup_t |X_t|$, we have, for $p > 1$,*

$$E(X^{*p}) \leq (\frac{p}{p-1}) \sup_t E(|X_t|^p).$$

Proof : Let μ be the law of X^*; then $E(X^{*p}) = \int_0^\infty \lambda^p d\mu$ and by an integration by parts we get by theorem (4.4.2), $E(X^{*p}) = \int_0^\infty p\lambda^{p-1}P(X^* \geq \lambda)d\lambda \leq \int_0^\infty p\lambda^{p-1}\frac{1}{\lambda}(\int_{|X_N|\geq\lambda} |X_N|dP)d\lambda$. To estimate the last integral we interchange the order of integration to get

$$E(X^{*p}) \leq pE(|X_N| \int_0^{|X_N|} \lambda^{p-2}d\lambda) \leq \ (\frac{p}{p-1})E(|X_N|^p).$$

4.4.2 Law of the iterated logarithm

Theorem 4.4.3. *Let B denote the standard real Brownian motion. Then, a.s.,*

$$\overline{\lim}_{t\to 0} \frac{B_t}{\sqrt{2t\ln(\ln\frac{1}{t})}} = 1 \tag{4.1}$$

Proof : It starts with the

Lemma 4.4.1. *The process $Y_{t,\alpha} = \exp(\alpha B_t - \alpha^2 t/2)$ is a martingale.*

Proof : $E[Y_{t,\alpha}|\mathcal{F}_s] = E[Y_{s,\alpha} \exp(\alpha(B_t - B_s) - \alpha^2(t-s)/2|\mathcal{F}_t] = Y_{s,\alpha)} E[Z_{t,s}|\mathcal{F}_s]$ and the result follows from the fact that Z is independent of $\mathcal{F}_s$ and that $E[Z] = 1$.
We define now $S_t = \sup\{B_s,\, s \leq t\}$:

Lemma 4.4.2. *For $a > 0$, $P[S_t > at] \leq \exp(-a^2t/2)$.*

Proof : We have $\exp(\alpha S_t - \alpha^2 t/2) = \sup_{s\leq t} Y_{s,\alpha}$ hence

$$P[S_t \geq at] \leq P[\sup_{s\leq t} Y_{s,\alpha} \geq \exp(\alpha a t - \alpha^2 t/2)] \leq \exp(-\alpha a t + \alpha^2 t/2) E[Y_{t,\alpha}]$$

by the maximal inequality. But $E[Y_{t,\alpha}] = E[Y_{0,\alpha}] = 1$ and $\inf_{\alpha>0}(-\alpha a t + \alpha^2 t/2) = -a^2t/2$ and the result follows.
We now come to the proof of the theorem : let $h(t) = \sqrt{2t\ln(\ln\frac{1}{t})}$ and $\theta,\, \delta \in (0,1)$. We define

$$\alpha_n = (1+\delta)\theta^{-n}h(\theta^n) \quad \beta_n = h(\theta^n)/2.$$

Using the same reasonning as in the previous lemmas, we get

$$P[\sup_{s\leq 1}(B_s - \alpha_n s/2) \geq \beta_n] \leq e^{-\alpha_n\beta_n} = Kn^{-(1+\delta)}$$

for some constant K. By Borel-Cantelli lemma, for almost every ω there exists $n_0(\omega)$ such that for $n \geq n_0(\omega), s \in [\theta^n, \theta^{n-1})$,

$$B_s(\omega) \leq \frac{\alpha_n\theta^{n-1}}{2} + \beta_n = [\frac{1+\delta}{2\theta} + \frac{1}{2}]h(\theta^n) \leq [\frac{1+\delta}{2\theta} + \frac{1}{2}]h(s).$$

As a result

$$\overline{\lim}_{s\to 0}\frac{B_s}{h(s)} \leq \frac{1+\delta}{2\theta} + \frac{1}{2} a.s.$$

and we get the $\leq$ inequality in the theorem by letting $\theta \to 1$, $\delta \to 0$.
For the proof of the opposite inequality we consider the events

$$A_n = \{B_{\theta^n} - B_{\theta^{n+1}} \geq (1-\sqrt{\theta})h(\theta^n)\}.$$

These events are independent and a striaightforward computation shows that

$$P(A_n) \geq \frac{a}{1+a^2}e^{-a^2/2}$$

with $a = (1-\sqrt{\theta})\sqrt{\frac{2\ln\ln\theta^{-n}}{1-\theta}}$ which makes $P(A_n)$ greater than $n^{-\gamma}, \gamma = (1 - 2\sqrt{\theta} + \theta)/(1-\theta) < 1$. By Borel-Cantelli lemma again we have that a.s.

$$B_{\theta^n} > (1-\sqrt{\theta})h(\theta^n) + B_{\theta^{n+1}}.$$

Since $-B$ is also a Brownian motion we know that $-B_{\theta^{n+1}}(\omega) < 2h(\theta^{n+1})$ from $n_0(\omega)$ on. it follows that $B_{\theta^n} > h(\theta^n)(1-5\sqrt{\theta})$ infinitely often, and the theorem is proven.

4.4.3 Optional Stopping Theorem

We recall that a family $(X_t)_{t\in T}$ of random variables is said to be uniformly integrable if

$$\forall \varepsilon > 0 \exists \delta > 0;\ \forall t \in T \forall E \in \mathcal{F}_t,\ P(E) < \delta \Rightarrow \int_E |X_t| dP < \varepsilon.$$

An important example of uniformly integrable family is that of a bounded family in L^p for some $p > 1$.

Theorem 4.4.4. *For a martingale $(X_t)_{t\in\mathbb{R}_+}$ the following three conditions are equivalent :*
1) (X_t) converges in L^1.
2) There exists a random variable $X_\infty \in L^1$ such that $\forall t \geq 0, X_t = E(X_\infty|\mathcal{F}_t$,
3) The family X_t is uniformly integrable.

Proof : The fact that 2) $\Rightarrow$ 3) is obvious. If 3) holds then in particular $\sup_t E(|X_t|) < +\infty$. Let us then show that a martingale satisfying this last property is converging a.s. Let f be a function $T \to \overline{\mathbb{R}}_+$, $t_1 < t_2 < .. < t_d$ a finite subset F of T : if $a < b$ are two reals we define $s_1 = \inf\{t_i;\ f(t_i) > b\}$, $s_2 = \inf\{t_i > s_1;\ f(t_i) < a\}$ and inductively $s_{2k+1} = \inf\{t_i > s_{2k};\ f(t_i) > b\}$, $s_{2k+2} = \inf\{t_i > s_{2k+1};\ f(t_i) < a\}(\inf(\emptyset) = t_d)$. We then put

$$D(f, F, [a, b]) = \sup\{n;\ s_{2n} < t_d\}$$

and define the downcrossing of $[a, b]$ by f as

$$D(f, [a, b]) = \sup_{F\subset T,\,\text{finite}} D(f, F, [a, b]).$$

Lemma 4.4.3. *If X is a martingale then*

$$\forall a < b,\ (b - a)E(D(X, [a, b]) \leq \sup_{t\in T} E\left[(X_t - b)^+\right].$$

Proof : We may assume that $T = F$ is finite. The $s_k's$ are now stopping times and $A_k = \{s_k < t_d\} \in \mathcal{F}_{s_k}$. Moreover $A_k \supset A_{k+1}$, $X_{s_{2n-1}} > b$ on A_{2n-1}, $X_{s_{2n}} < a$ on A_{2n}. Therefore, by corollary (4.4.1)

$$0 \leq \int_{A_{2n-1}} (X_{s_{2n-1}} - b) dP \leq \int_{A_{2n-1}} (X_{s_{2n}} - b) dP \leq (a - b)P(A_{2n}) + \int_{A_{2n-1}\setminus A_{s_{2n}}} (X_{2n} - b) dP.$$

Consequently, since $s_{2n} = t_d$ on the complement of A_{2n},

$$(b - a)P(A_{2n}) \leq \int_{A_{2n-1}\setminus A_{2n}} (X_{t_d} - b)^+ dP.$$

But $A_{2n} = \{D(X, T, [a, b]) > n\}$ and the sets $A_{2n-1} \backslash A_{2n}$ are disjoint : the result then follows by adding these inequalities.
Recall that we want to prove that if $\sup_t E(|X_t|) < +\infty$ then X_t is a.s. converging as $t \to \infty$. If this were not the case then there would exist $a < b$ such that $\underline{\lim}_{t\to\infty}(X_t) < a < b < \overline{\lim}_{t\to\infty}(X_t)$ on a set of positive probability. But this would imply that $D(X, [a, b]) = +\infty$ on this set, which is impossible by the preceeding lemma. With the use of classical measure theory, the implication 3)$\Rightarrow$ 1) is thus proven. The fact that 1)$\Rightarrow$ 2) follows by passing to the limit as $s \to \infty$ in the equality

$$X_t = E(X_{t+s}|\mathcal{F}_t).$$

Theorem 4.4.5. *(Optional stopping theorem) If X is a martingale and if S, T are two bounded stopping times with $S \leq T$ then*

$$X_S = E[X_T|\mathcal{F}_S]. \tag{4.2}$$

If X is uniformly integrable, the family (X_S) where S runs through the set of all stopping times is uniformly integrable and if $S \leq T$,

$$X_S = E[X_T|\mathcal{F}_S] = E[X_\infty|\mathcal{F}_S]. \tag{4.3}$$

Proof : It suffices to prove (4.3) because a matingale defined on a closed interval is uniformly integrable. It is true if S, T are bounded by (4.4.1) and the result follows by approximation.

The preceeding theorem is false if the martingale is not assumed to be uniformly integrable. To see this, consider a positive martingale going to 0, (for example $X_t = \exp(B_t - t/2)$ where B_t is a usual Brownian, $X_0 = 1$) : if $T = \inf\{t \geq 0;\ X_t \leq \alpha\}$ then $E[X_T] = \alpha \neq E[X_0] = 1$.

4.5 Stochastic Integration.

4.5.1 Quadratic Variations.

Definition 4.5.1. *A process A is called of finite variation if it is adapted and if the paths $t \mapsto A_t(\omega)$ are right-continuous and of bounded variation.*

If X is a progressively measurable process (i.e. if for every t the map $(s, \omega) \mapsto X_s(\omega)$ is measurable on $[0, t] \times \Omega$) and bounded on every interval $[0, t]$ then one can define

$$(X \cdot A)_t = \int_0^t X_s(\omega) dA_s(\omega).$$

We aim to define a similar integral for martingales A. This cannot be defined as before because of the

Proposition 4.5.1. *If M is a continuous martingale of bounded variation then M is constant.*

Proof : Let $t_1, ..., t_n$ be a subdivision of $[0, t]$. Then if we assume that $M_0 = 0$ we have

$$E[M_t^2] \leq E\left[\sum_{i=0}^{n-1}(M_{t_{i+1}}^2 - M_{t_i}^2)\right] = E\left[\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2\right] \leq V \sup_i |M_{t_{i+1}} - M_{t_i}| \to 0$$

as the mesh goes to 0. This means that one cannot proceed to a path by path integration. Instead we are going to use a more global method and the notion of quadratic variation.
If $\Delta = \{t_0 < ... < t_k < ..\}$ is a subdivision of $\mathbb{R}_+$ we define its modulus as $\sup\{t_{k+1} - t_k, \, k \geq 0\}$ and, if M is a process, we define, for $t \geq 0$,

$$T_t^\Delta = \sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_n})^2$$

where n is such that $t_n \leq t < t_{n+1}$.

Definition 4.5.2. *We say that a process M is of finite quadratic variation if there exists a process denoted by $< M, M >$ such that T_t^Δ converges in probability towards $< M, M >$ as the modulus of Δ goes to* 0.

Theorem 4.5.1. *A continuous and bounded martingale M is of finite quadratic variation. Moreover, $< M, M >$ is the unique continuous increasing adapted process vanishing at* 0 *such that $M^2 - < M, M >$ is a martingale.*

Proof : We only outline it. We first easily see that if Δ is a subdivision then $M^2 - T^\Delta$ is a continuous martingale. It thus remains only to show that if Δ_n is a sequence of subdivisions of the interval $[0, a]$ whose modulus converges to 0 then $T_a^{\Delta_n}$ converges in L^2. We have thus to show that if $|\Delta| + |\Delta'| \to 0$ then $E[|T_a^\Delta - T_a^{\Delta'}|^2] \to 0$. We complete the proof in the case Δ' is Δ completed by a point s_i in each interval $[t_i, t_{i+1}]$: then

$$|T_a^\Delta - T_a^{\Delta'}| = 2(M_{t_i} - M_{s_i})(M_{t_{i+1}} - M_{s_i})$$

and thus $E[|T_a^\Delta - T_a^{\Delta'}|^2] \leq 4E[\sup |M_{t_{i+1}} - M_{s_i}|^4]^{1/2} E[(T_a^{\Delta'})^2]^{1/2}$ and it is sufficient to prove that $E[(T_a^{\Delta'})^2]$ remains bounded as the modulus goes to 0.
In order to prove this we write

$$(T_a^\Delta)^2 = (\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2)^2$$

$$= 2\sum_{k=0}^{n-1}(T_a^\Delta - T_{t_k}^\Delta)(T_{t_{k+1}}^\Delta - T_{t_k}^\Delta) + \sum_{k=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^4.$$

But

$$E[T_a^\Delta - T_{t_k}^\Delta|\mathcal{F}_k] = E[(M_a - M_{t_k})^2|\mathcal{F}_k]$$

and thus

$$E[(T_a^\Delta)^2] = 2\sum_{k=0}^{n-1} E[(M_a - M_{t_k})^2(T_{t_{k+1}}^\Delta - T_{t_k}^\Delta) + \sum_{k=0}^{n-1} E[(M_{t_{i+1}} - M_{t_i})^4]$$

$$\leq 12C^2 E[T_a^\Delta] \leq 48C^4$$

where C is a bound for the martingale M.

This theorem is very interesting but its hypothesises are very strong. It does not cover for instance the case of the Brownian motion (a non-uniformly integrable martingale) though Brownian motion has a quadratic variation, namely $B_t^2 - t$ is a martingale. In order to cover this case we need the notion of local martingale.

Definition 4.5.3. *An adapted right continuous process X is called a local martingale if there exists stopping times T_n, $n \geq 0$ increasing to $+\infty$ a.s. such that for every n the process $X^{T_n}1_{[T_n>0]}$ is a uniformly integrable martingale.*

In this statement we have used the notation $X^T = X_{T\wedge t}$. If the process X is continuous we can further use the stopping time $S_n = \inf\{t > 0; |X_t| = n\}$ and replace T_n by $T_n \wedge S_n$, meanning that we can assume that the martingale $X^{T_n}1_{[T_n>0]}$ is bounded.
We may now state the general

Theorem 4.5.2. *If M is a continuous local martingale there exists a unique continuous increasing process $< M, M >$ such that $M^2- < M, M >$ is a continuous local martingale.*

To prove this thorem we use a sequence T_n of stopping times increasing to ∞ such that for all n, $X_n = X^{T_n}1_{[T_n>0]}$ is a bounded martingale. By the theorem for bounded martingales there exists an increasing process A_n such that $X_n^2 - A_n$ is a bounded martingale. It is easy to see that $A_{n+1}^{T_n} = A_n$ on $[T_n > 0]$ and we can thus define unambiguously $< M, M >$ by setting it to be equal to A_n on $[T_n > 0]$. This process is the one we were looking for.
The next theorem generalizes the preceeding in the sense that it polarizes it :

Theorem 4.5.3. *If M, N are two continuous local martingales there exists a unique process $< M, N >$ with bounded variation, vanishing at 0, such that $MN- < M, N >$ is a local martingale.*

Proof : $< M, N >= \frac{1}{4}[< M + N, M + N > - < M - N, M - N >]$.
Remark : It is an easy exercise to show that if $\sigma(M_s, s \leq t)$ is independent of $\sigma(N_s, s \leq t)$ then MN is still a martingale, showing that $< M, N >= 0$ in this case.

Theorem 4.5.4. *If M, N are two local martingales and H, K are two measurable processes then, a.s. for all $t \leq \infty$,*

$$\int_0^t |H_s||K_s||d < M, N >_s | \leq (\int_0^t |H_s|^2 |d < M, M >_s |)^{1/2} (\int_0^t |K_s|^2 |d < N, N >_s |)^{1/2} \tag{4.4}$$

Proof : It suffices to prove the theorem for processes of the form

$$K = K_0 1_0 + K_1 1_{]0,t_1]} + ... + K_n 1_{]t_{n-1},t_n]}.$$

We now define $< M, N >_s^t = < M, N >_t - < M, N >_s$. Since almost surely for every $r \in \mathbb{Q}$ we have

$$< M, M >_s^t + 2r < M, N >_s^t + r^2 < N, N >_s^t = < M + rN, M + rN >_s^t \geq 0,$$

we must have

$$| < M, N >_s^t | \leq (< M, M >_s^t)^{1/2} (< N, N >_s^t)^{1/2} a.s.$$

As a result,

$$| \int_0^t H_s K_s d < M, N >_s | \leq \sum_i |H_i K_i| | < M, N >_{t_i}^{t_{i+1}} |$$
$$\leq \sum_i |H_i K_i| (< M, M >_{t_i}^{t_{i+1}})^{1/2} (< N, N >_{t_i}^{t_{i+1}})^{1/2}$$

and the result follows by application of Cauchy-Schwarz inequality.

Corollary 4.5.1. *(Kunita Watanabe inequality) If $1/p + 1/q = 1$, $p \geq 1$, then*

$$E[\int_0^\infty |H_s||K_s||d < M, N >_s] \leq \|(\int_0^\infty |H_s|^2 |d < M, M >_s |)^{1/2}\|_p \|(\int_0^\infty |K_s|^2 |d < N, N >_s |)^{1/2}\|_q \tag{4.5}$$

We now introduce the important (Hardy) space H^2, the space of L^2 martingales. We have already seen that this space is in a natural one to one correspondance with L^2. Thus H^2 is a Hilbert space for the norm

$$\|M\|_{\mathbb{H}^2} = E[M_\infty^2]^{1/2}.$$

The subspace H_0^2 consists of those martingales in H^2 such that $M_0 = 0$.

Theorem 4.5.5. *A continuous local martingale M is in H^2 if and only if $M_0 \in L^2$ and $E[< M, M >_\infty] < \infty$.*

Proof : Let T_n be a sequence of stopping times such that $M^{T_n} 1_{[T_n > 0]}$ is bounded. We can write

$$E[M_{T_n \wedge t}^2 1_{[T_n > 0]}] - E[< M, M >_{T_n \wedge t} 1_{[T_n > 0]}] = E[M_0^2 1_{[T_n > 0]}]$$

and the result follows by passing to the limit as $n \to \infty$.

4.6 Stochastic Integration

For reasons that will appear clearly later we need a notion of integration along brownian paths. But this cannot be done naively since Brownian motion is not of bounded variation : Riemann sums do not converge pathwise but will be shown to converge in probability. Before we come to this point we define integration with respect to the elements of H^2.

Definition 4.6.1. *if $M \in H^2$ we define $\mathcal{L}^2(M)$ the space of progressively measurable processes K such that*

$$\|K\|_M^2 = E[\int_0^\infty K_s^2 d < M, M >_s] < +\infty.$$

We can define a bounded measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ by putting

$$P_M(\Gamma) = E[\int_0^\infty 1_\Gamma(s,\omega) d < M, M >_s (\omega)$$

and $\mathcal{L}^2(M)$ appears as the space of P_M-square integrable, progressively measurable functions and we can then define as usual the Hilbert space $L^2(M)$.

Theorem 4.6.1. *Let $M \in H^2$: for each $K \in H^2$ there exists a unique element of H_0^2, denoted by $K \cdot M$ such that for every $N \in H^2$*

$$< K \cdot M, N >= K \cdot < M, N >$$

(notice that the two $\cdot$ have a different meanning). Moreover the map $K \mapsto K \cdot M$ is an isometry between $L^2(M)$ and H_0^2.

Proof : Uniqueness is obvious. To prove existence we observe, by Kunita-Watanabe inequality, that for every $N \in H_0^2$ we have

$$|E[\int_0^\infty K_s d < M, N >_s]| \leq \|N\|_{H^2} \|K\|_M$$

which implies that the map $N \mapsto E[(K \cdot < M, N >)_\infty]$ is a continuous linear form on the Hilbert space H_0^2 . There is thus an element $K \cdot M \in H_0^2$ such that

$$\forall N \in H_0^2,\ E[(K \cdot M)_\infty N_\infty] = E[(K \cdot < M, N >)_\infty].$$

Let T be a stopping time ; me may write

$$\begin{aligned} E[(K \cdot M)_T N_T] &= E[E[(K \cdot M)_\infty | \mathcal{F}_T] N_T] = E[(K \cdot M)_\infty N_T] \\ &= E[(K \cdot M)_\infty N_\infty^T] = E[(K \cdot < M, N^T >)_\infty] \\ &= E[(K \cdot < M, N >^T)_\infty] = E[(K \cdot < M, N >)_T] \end{aligned}$$

which proves that $(K \cdot M)N - K \cdot < M, N >$ is a martingale and thus the first result. The fact that $K \mapsto K \cdot M$ is an isometry is obvious. Finally in the general case $M \in H^2$ we simply set $K \cdot M = K \cdot (M - M_0)$ and all the properties are easily checked.

Definition 4.6.2. *The martingale $K \cdot M$ is the Ito integral or stochastic integral of K wrt M and is also denoted by*

$$(K \cdot M)_t = \int_0^t K_s dM_s.$$

Let $\mathcal{E}$ be the space of elementary processes, i.e. processes of the form

$$K = K_{-1}1_0 + \sum_i K_i 1_{]t_i,t_{i+1}]}$$

where (t_i) is a sequence increasing to $+\infty$. In this case it is not hard to see that

$$(K \cdot M)_t = \sum_{i=0}^{n-1} K_i(M_{t_{i+1}} - M_{t_i}) + K_n(M_t - M_{t_n})$$

whenever $t \in [t_n, t_{n+1}[$. the following theorem is left as an exercise to the reader :

Theorem 4.6.2. *If $K \in L^2(M), H \in L^2(K \cdot M)$,then $HK \in L^2(M)$ and*

$$(HK) \cdot M = H \cdot (K \cdot M).$$

Now we want to define a stochastic integral wrt general local martingales, the main purpose being integration wrt the Brownian. For this purpose we introduce the

Definition 4.6.3. *If M is a continuous local martingale we call $L^2_{loc}(M)$ the space of progressively measurable processes K for which there exists a sequence of stopping times T_n increasing to ∞ such that*

$$E[\int_0^{T_n} K_s^2 d < M, M >_s] < +\infty.$$

Contrarily as it may seem at a first glance, this notion is very general. It englobes for instance all locally bounded processes and thus, in particular, all continuous processes.

Theorem 4.6.3. *For any $K \in L^2_{loc}(M)$ there exists a unique continuous local martingale denoted $K \cdot M$ such that for any continuous local martingale N,*

$$< K \cdot M, N >= K \cdot < M, N > .$$

Proof : One can choose a sequence of stopping times T^n such that $M^{T_n} \in H^2$ and $K^{T_n} \in L^2(M^{T_n})$ and thus define $X^{(n)} = K^{T_n} \cdot M^{T_n}$.

Lemma 4.6.1. *If T is a stopping time,*

$$K \cdot M^T = K1_{[0,T]} \cdot M = (K \cdot M)^T.$$

The proof is left to the reader.
This lemma implies that $X^{(n+1)} = X^{(n)}$ on $[0, T_n]$. This defines unambiguously a process $K \cdot M$ and all the properties are easily derived.

4.7 Itô's formula

From now on we will call semimartingale any process that can be expressed as a sum of a local martingale and a process of finite variation. If X is a continuous semimartingale, for which functions F of a real variable is it true that $F(X)$ is still a semimartingale ? Itô's formula will in particular give an answer to this question. We start with the special case $F(x) = x^2$.

Proposition 4.7.1. *If X, Y are continuous semimartingales then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + < X, Y >_t .$$

Proof : The case $X = Y$ follows almost immediately from the obvious formula :

$$\sum_i (X_{t_{i+1}} - X_{t_i})^2 = X_t^2 - X_0^2 - 2 \sum_i X_{t_i}(X_{t_{i+1}} - X_{t_i}).$$

The general case is obtained by the usual polarization.

Notice that in the case where X is a local martingale, we already know that $X^2 - < X, X >$ is a local martingale : Itô's formula gives a formula for this local martingale. In the case X is of finite variation, Itô's formula reduces to the ordinary integration by parts. In the case of Brownian motion, Itô's formula reads

$$B_t^2 - t = 2 \int_0^t B_s dB_s.$$

We now come to the famous Itô formula. In order to state it in a sufficient generality we introduce the notion of d-dimensional vector local (continuous semi) martingale. It is a $\mathbb{R}^d$ valued process $X = (X_1, ..., X_d)$ such that each of its components is a local (continuous semi) martingale.

Theorem 4.7.1. *(Itô's formula) Let $F : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function and X a continuous vector semimartingale ; then $F(X)$ is a continuous semimartingale and*

$$F(X_t) =$$

$$F(X_0) + \sum_i \int_0^t \frac{\partial F}{\partial x_i}(X_s) dX_{i,s} + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 F}{\partial x_i x_j}(X_s) d < X_i, X_j >_s .$$

Proof : We outline it in the case $d = 1$.Suppose that Itô's formula is valid for the function F and let us consider the function $G = xF$. Then by (**??**) we have

$$G = G(X_0) + X \cdot F(X) + F(X) \cdot X + < X, F(X) > .$$

On the other hand, since F satisfies Itô's formula

$$F(X) = F(X_0) + F'(X) \cdot X + F(X) \cdot X + \frac{1}{2} F''(X) \cdot < X, X > .$$

If we replace $F(X)$ by this expression we obtain

$$X \cdot F(X) = X \cdot (F'(X) \cdot X) + \frac{1}{2} X \cdot (F''(X) \cdot < X, X >= (XF'(X)) \cdot X + \frac{1}{2} XF''(X) \cdot < X, X >$$

Similarly,

$$< X, F(X) >=< X, F'(X) \cdot X > + \frac{1}{2} < X, F''(X) \cdot < X, X >>$$

$$= F'(X) < X, X > + \frac{1}{2} F''(X) < X, < X, X >>= F'(X) < X, X > .$$

On the other hand

$$G(X_0) + G'(X) \cdot X + \frac{1}{2} G''(X) \cdot < X, X >$$

$$= G(X_0) + XF'(X) \cdot X + F'(X) \cdot < X, X > + \frac{1}{2} XF''(X) \cdot < X, X >,$$

and we get that Itô's formula is valid for G. It follows that its is valid for all polynômials ; an easy approximation argument then implies that it is valid for any C^2 function.
We state an first important consequence of this formula :

Theorem 4.7.2. *If f is a complex function defined on $\mathbb{R} \times \mathbb{R}_+$ of class C^2 and satisfying the heat equation*

$$\frac{\partial f}{\partial y} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0,$$

then for any continuous local martingale M the process $f(M, < M, M >)$ is a local martingale. In particular the process

$$\mathcal{E}^\lambda(M) = \exp\{\lambda M_t - \frac{\lambda^2}{2} < M, M >_t\}$$

is a local martingale. If $\lambda = 1$ we speak of this process as the exponential of M.

Proof : Itô's formula gives, writing $N =< M, M >$, that

$$f(M, N)_t = f(M, N)_0 + \int_0^t \frac{\partial f}{\partial x} dM_s + \int_0^t \frac{\partial f}{\partial y} dN_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2} d < M, M >_s = \int_0^t \frac{\partial f}{\partial x} dM_s$$

4.8 Martingales as Time-changed Brownian Motion

Theorem 4.8.1. *(Paul Lévy) For a continuous adapted d-dimensional process X vanishing at 0 the following three conditions are equivalent :*

- *(i) X is a Brownian Motion.*
- *(ii) X is a continuous local martingale and $< X^i, X^j >_t = \delta_{ij} t$, $1 \leq i, j \leq d$.*
- *(iii) X is a continuous local martingale and for any d-uple $f_1, ..., f_d$ of $L^2(\mathbb{R}_+)$ functions the process*

$$\mathcal{E}_t^{if} = \exp\left\{ i \sum_k \int_0^t f_k(s) dX_s^k + \frac{1}{2} \sum_k \int_0^t f_k(s)^2 ds \right\}$$

is a complex local martingale.

Proof : (in the case $d = 1$).(i)$\Rightarrow$(ii) is known already. The fact that (ii)$\Rightarrow$ (iii) follows from theorem (4.7.2) applied with $\lambda = i, dM = f dX$. Suppose finally that (iii) holds : we apply it with $f = \xi 1_{[0,T]}$ and it gives that the process

$$\mathcal{E}_t^{if} = \exp\left\{ i\xi X_{t\wedge T} + \frac{1}{2}\xi^2 t \wedge T \right\}$$

is a martingale. For $A \in \mathcal{F}_s$, $s < t < T$ we get

$$E[1_A \exp\{i\xi(X_t - X_s)\}] = P(A) \exp\left\{ \left(-\frac{\xi^2}{2}(t-s) \right) \right\},$$

which implies that $X_t - X_s$ is independent of $\mathcal{F}_s$ and has a Gaussian distribution with variance $(t-s)$; hence (i) holds.

We now come to the fundamental characterization of martingales. For this purpose we need the notion of time-change. Consider a right continuous increasing adapted process A ; we can associate to this process the stopping times $C_s = \inf\{t; A_s > t\}$. The reader is encouraged to check that (C_s) is a right-continuous process and that the filtration $\mathcal{F}_{C_s}$ is also right continuous. Moreover, for any t, the random variable A_t is a $(\mathcal{F}_{C_s})$-stopping time.

Definition 4.8.1. *A time-change is a family of stopping times $(C_s), s \geq 0$ such that a.s. $s \mapsto C_s$ is increasing and riht-continuous.*

If C is a time-change and X is a progressive process we define $\hat{X}_t = X_{C_t}, \hat{\mathcal{F}}_t = \mathcal{F}_{C_t}$. The process $\hat{X}$ is called the time-changed process of X.

We want to prove that the class of semimartingales is stable under this operation. We formally prove that $\hat{X}$ is a local martingale if X, from which the result follows. So let X be a local martingale and T a stopping time such that X^T is bounded. The time $\hat{T} = \inf\{t; C^t \geq T\}$ is a $\hat{\mathcal{F}}_t$-stopping time and $\hat{X}_t^{\hat{T}} = X_{C_t}^T$. By the optional stopping theorem $\hat{X}^{\hat{T}}$ is a martingale. Considering sequences of such stopping times we obtain that $\hat{X}$ is a local martingale.

Theorem 4.8.2. *(Dambis,Dubins-Schwarz). If M is a continuous local martingale vanishing at 0 and such that $< M, M >_\infty = \infty$ then, if we set*

$$T_t = \inf\{s :< M, M >_s > t\},$$

$B_t = M_{T_t}$ is a $\mathcal{F}_{T_t}$-Brownian motion and $M_t = B_{<M,M>_t}$.

Proof : By the result outlined before the theorem B is a continuous local $(\mathcal{F}_{T_t})$-martingale and $< B, B >_t = < M, M >_{T_t} = t$. it is thus a Brownian motion by Paul Lévy's characterization.

Chapitre 5

Stochastic Löwner Evolution

5.1 Bessel Processes

We start by considering standard Brownian motion in $\mathbb{R}^d$, i.e. $B = (B_1, ..., Bd)$. We denote by R the process $R = \|B\| = \sqrt{B_1^2 + ... + B_d^2}$. If we apply Itô's formula we get

$$dR = \frac{\frac{d-1}{2}}{R} dt + \sum_{d}^{j=1} \frac{B_j}{R} dB_j$$

But $M = \sum_d^{j=1} \frac{B_j}{R} dB_j$ is a local martingale with $< M, M >_t = t$ so that it is a Brownian motion. This motivates the

Definition 5.1.1. *For $x > 0$ we define a Bessel $d-$process as a solution of the stochastic differential equation (SDE)*

$$dX_t^x = \frac{a}{X_t^x} dt + dB_t, X_0^x = x$$

where a=(d-1)/2.

If we solve the above SDE, it is understood that we take the same ω for different values of x. It follows that if $x < y$ then $X_t^x < X_t^y$ (by uniqueness of solution) for all values of t less than T_x, the life-time of X_t^x, that is

$$T_x = \sup\{t > 0\,;\, X_t^x > 0\}$$

This implies in particular that $T_x \leq T_y$.
It will be useful to notice the scaling law of Bessel processes $\frac{1}{x} X_{x^2 t}^x \approx X_t^x$, $\approx$ meanning having the same law.
The following theorem shows the different phases of Bessel processes that will reflect in the different phases of SLE later on :

Theorem 5.1.1. *According to the value of a, we have :*

1. *If $a \geq 1/2$, then for all $x > 0, T_x = +\infty$ a.s. and $\overline{\lim}_{t\to\infty} X_t^x = +\infty a.s.$*
2. *If $a = 1/2$ then $\inf_{t>0} X_t^x = 0$ a.s.*
3. *If $a > 1/2$ then for all $x > 0, X_t^x \longrightarrow \infty$ a.s.*
4. *If $a < 1/2$ then for all $x > 0, T_x < \infty$ a.s.*
5. *If $1/4 < a < 1/2$, $x < y$ then $P(T_x = T_y) > 0$.*
6. *If $a \leq 1/4\,, x < y$, then $T_x < T_y$ a.s.*

Proof : Let $0 < x_1 < x_2$ be fixed numbers and consider $x \in [x_1, x_2]$. We define $\sigma = \inf\{t > 0\,;\, X_x^t \in \{x_1, x_2\}\}$ and $\Phi(x; x_1, x_2) = P(X_\sigma^x = x_2)$. It is obvious that

$$\Phi(X_{t\wedge\sigma}^x) = E\left[\Phi(X_\sigma^x)|\mathcal{F}_t\right]$$

and hence that $\Phi(X_{t\wedge\sigma}^x)$ is a martingale. It follows that the drift term in Itô formula must vanish and this reads

$$\frac{1}{2}\Phi''(x) + \frac{a}{x}\Phi'(x) = 0.$$

Knowing that $\Phi(x_1) = 0, \Phi(x_2) = 1$ we have the formulas

$$\Phi(x) = \frac{x^{1-2a} - x_1^{1-2a}}{x_2^{1-2a} - x_1^{1-2a}},\ a \neq \frac{1}{2},$$

$$\Phi(x) = \frac{\ln(x) - \ln(x_1)}{\ln(x_2) - \ln(x_1)},\ a = \frac{1}{2}.$$

We start with the properties of the case $a \geq 1/2$:
First of all

$$\lim_{x_1\to 1} \Phi(x; x_1, x_2) = 1$$

in this case. It follows immediately that for all $x_2 > 0$, X_t^x will reach x_2 before 0. The second part of the fist point follows. To prove the first it suffices to see that X_t^x cannot reach ∞ in finite time. To see this last point consider T_n the first arrival at 2^n and S_n the greatest $t \leq T_{n+1}$ such that $X_t^x = 2^n$. Then it is easy to see that the expectation of $T_{n+1} - S_n$ is greater than $c4^n$ and an easy argument using Borel-Cantelli lemma allows to conclude.
The second point follows from the fact that

$$\lim_{x_2\to+\infty} \Phi(x; x_1, x_2) = 0$$

if $a = 1/2$, from which it follows that for every $x_1 > 0$ there exists $M > 0$ such that X_t^x will reach x_1 before M with probability 1. The second point follows.
We come to the third point : we already know that $\overline{\lim} X_t^x = +\infty$. Let T_n the first passage to 2^n. We have

$$\lim_{x_2\to+\infty} \Phi(x; x_1, x_2) = 1 - (\frac{x_1}{x})^{2a-1} = l.$$

More precisely

$$|\Phi(x;x_1,x_2)-l|\leq(x^{1-2a}-x_2^{1-2a})(\frac{x_1}{x_2})^{2a-1}$$

and we deduce from this inequality that the probability that between T_n and T_{n+1} the process reaches $2^n/M_n$ is less than C/M_n^{2a-1}. Taking

$$M_n=n^{\frac{2}{2a-1}}$$

we conclude with Borel-Cantelli.
For the rest of the proof we assume $a<1/2$: we have

$$\Phi(x;0,x_2)=(\frac{x}{x_2})^{1-2a}\rightarrow 0,\ x_1\rightarrow 0,$$

and thus a.s. there exists $x_2>0$ such that X_t^x reaches 0 before x^2. This proves the forth point.
We come to the proof of the $5th$ point : we already know that $T_x\leq T_y<+\infty$. Put $q(x,y)=P(T_x=T_y)$: by scaling, it is obvious that $q(x,y)=q(1,y/x)$.

Lemma 5.1.1. *For all fixed $t>0, \lim_{r\longrightarrow\infty}P(T_r<t)=0$.*

Proof : A small computation using Itô shows that

$$X_t^r-r=(2a+1)t+\int_0^t 2X_s^r dB_s$$

so that

$$-r=(2a+1)T_r+\int_0^{T_r}2X_s dB_s$$

and the result follows by Tchebychev inequality.
As a corollary, $\lim_{r\longrightarrow\infty}q(1,r)=0$.

Lemma 5.1.2. *The event $\{T_1=T_y\}$ is equal (up to a set of probability 0) to the set*

$$\left\{\sup_{t<T_1}\frac{X_t^y-X_t^1}{X_t^1}<+\infty\right\}.$$

Proof : It is obvious that the last statement implies that $T_1=T_y$. Conversely, by the strong Markov property,

$$P\left\{T_y=T_1\,;\sup_{t>0}\left\{\frac{X_t^y-X_t^1}{X_t^1}\right\}\geq r\right\}\leq q(1,1+r),$$

which goes to 0 as r goes to ∞.
Let $Z_t=\ln(\frac{X_t^y-X_t^1}{X_t^1})$. By Itô's formula,

$$dZ_t=\left[(\frac{1}{2}-2a)\frac{1}{X_t^2}+a\frac{X_t^y-X_t^x}{X_t^yX_t^{x2}}\right]dt-\frac{1}{X_t^1}dB_t.$$

Define a time-change $r(t)$ by $\int_0^{r(t)}\frac{ds}{X_s^{1\,2}}=t$:

Lemma 5.1.3. $I = r^{-1}(T_x) = +\infty$.

Proof : It suffices to show that

$$\int_0^{T_x} = +\infty.$$

to do so we assume that $x = 1$ and denote by T_j the first arrival at 2^{-j}. We also put

$$Y_j = \int_{T_{j-1}}^{T_j} \frac{ds}{X_s^2} :$$

Then $I = \sum Y_j = +\infty$ a.s. because the variables Y_j are independent with the same distribution (by scaling) and with positive expectation.
Let $\tilde{Z}(t) = Z_{r(t)}$. Then $\tilde{Z}_t$ satisfies

$$d\tilde{Z}_t = \left[(\frac{1}{2} - 2a) + a\frac{X^y_{r(t)} - X^1_{r(t)}}{X^1_{r(t)}} \right] dt + d\tilde{B}_t,$$

where $\tilde{B}_t = -\int_0^{r(t)} X_s^{1\,-1} dB_s$ is a standard Brownian motion. After integration, we obtain

$$\tilde{Z}_t = \tilde{Z}_0 \tilde{B}_t + (\frac{1}{2} - 2a)t + a\int_0^t \frac{X^y_{r(s)} - X^1_{r(s)}}{X^1_{r(s)}} ds.$$

If $a \leq 1/4$ then $\tilde{Z}_t$ takes arbitrarily large values; by the preceeding discussion, we get point 5).
Suppose finally that $1/4 < a < 1/2$: choose $b \in (1/4, a)$ and let $\varepsilon = 2(a-b)/a$. Suppose $x = 1, y = 1 + \varepsilon/2$ and let σ be the first time that $X^y_{r(s)} - X^1_{r(s)} = \varepsilon X^1_{r(s)}$. For $0 \leq T_1 \wedge \sigma, \tilde{Z}_t \leq \tilde{Z}^*_t$ where

$$d\tilde{Z}^*_t = (\frac{1}{2} - 2b)dt + d\tilde{B}_t$$

Since $1/2 - 2b < 0$ there is a positive probability that $\tilde{Z}^*_t$ never reaches $\ln \varepsilon$ starting at $\ln \frac{\varepsilon}{2}$. On this event the same occurs for $\tilde{Z}_t$, which implies (5.1.2). We have thus shown that $q(1, 1 + \varepsilon/2) > 0$; since $X^y_{r(s)} - X^1_{r(s)}$ decreases with t, it follows easily that $q(x, y) > 0$ for all $0 < x < y$.

5.2 Definitions for SLE

We want to define a Löwner process having certain properties. This process will be define by a driving function $U_t, t \geq 0$ which is a continuous real random process. We recall that this means that we consider the differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, g_0(z) = z$$

and the growing family of sets K_t is then defined as the set of initial values having a life-time $\leq t$.The mapping g_t can then be seen as the Riemann mapping from $\mathcal{H}\backslash K_t$ with the hydrodynamic normalization $g_t(z) = z + ..$ at ∞. If $s < t$ we define $g_{s,t} = g_t \circ g_s^{-1}$ and $\overline{g}_{s,t}(z) = g_{s,t}(z + U_s) - U_s$. The choice of the driving function will be done in order that :

1. the distribution of $\overline{g}_{s,t}$ depends only on $t - s$,
2. Markovian property :$\overline{g}_{s,t}$ is independent of g_r, $r \leq s$.
3. the distribution of K_t is symmetric wrt the imaginary axis.

It is an exercise to see that the only possibility for the driving function is $U_t = \lambda B_t$ for some positive constant λ, B_t being a standard $1D$ Brownian motion. For reasons that will become clear later we set $\lambda = \sqrt{\kappa}$ and set the

Definition 5.2.1. *The chordal stochastic Löwner evolution with parameter $\kappa \geq 0(SLE_\kappa)$ is the random collection of conformal maps g_t solving the ODE*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t},\ g_0(z) = z.$$

An easy but important property of SLE is its scaling :

Proposition 5.2.1. *If g_t is a SLE_κ then it is the same for $\tilde{g}(z) = r^{-1} g_{r^2 t}(rz)$ and if γ is a $SLE-$ path, the same is true for $\tilde{\gamma}(t) = r^{-1}\gamma(r^2 t)$.*

5.3 SLE paths.

We recall that, since U_t is continuous, the corresponding increasing family of sets K_t is continuously growing. However the sole continuity does not warranty that this family is generated by a curve, i.e. that there exists a path γ such that for $t \geq 0, K_t$ is the unbounded component of $\mathcal{H}\backslash\gamma([0,t])$. However, in the case of SLE :

Theorem 5.3.1. *For every $\kappa \geq 0$ SLE_κ is generated by a curve.*

We are going to give a proof of this theorem for $\kappa \neq 8$. But this proof is long and difficult : we thus prefer to insist more on the scheme of the proof than on the technical details for which we invite the reader to consult the original proof [RS]. We begin by extending g_t to negative values of t by considering a two-sided Brownian motion thus defined on $\mathbb{R}$. We recall that $f_t = g_t^{-1}$: it is immediate that $g_{-t}(z)$ has the same law as $f_t(z+\xi(t))-\xi(t)$ where we have put $\xi(t) = \sqrt{\kappa}B_t$. We also define

$$\hat{f}_t(z) = f_t(z + \xi(t)).$$

We also notice that $\Im g_t(z)$ is deacreasing in time, allowing the time change

$$T_u(z) = \sup\{t \in \mathbb{R}\,;\, \Im g_t(z) \geq e^u\}.$$

Lemma 5.3.1. $\forall z \in \mathbb{H}\,,\, u \in \mathbb{R},\, T_u \neq \pm\infty.$

In other words $\lim \Im g_t(z) \to +\infty$ as $t \to -\infty$ and converges to 0 as t reaches the life-time of z.
Proof : Define $\overline{\xi}(t) = \sup\{|\xi(s)|\,,\, s \leq t\}$ and let $U = \{s \leq t\,;\, |g_s(z)| > \overline{\xi}(t)\}$ which is a union of disjoint intervals. Put $y = |g_s| - \xi(t)$ so that $yy' \leq 2$. Integrating this equation over a component of U we see that (taking in account the fact that the left element of the interval may be equal to 0,

$$|g_t(z)| \leq \overline{\xi}(t) + 2\sqrt{t} + |z|.$$

Putting $y_t = \Im g_t(z)$ we then deduce that

$$-\frac{\partial}{\partial t}\log y_t \geq \frac{2}{(\overline{\xi}(t) + 2\sqrt{t} + |z|)^2}.$$

By the law of the iterated logarithm, the right-hand side is not integrable over $[0, +\infty[$, a fact that implies the lemma.
The following theorem is the fondamental estimate : it gives the derivative estimates that go beyond the theorem we are proving. We wish to estimate $E[|g_t'(z)|^a]$. As we will see we will be able to approach only a related quantity which happens to be as useful. Let us first fix some notations :
We fix $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$ and if $u \in \mathbb{R}$ we set $z(u) = g_{T_u(\hat{z})}(\hat{z}) - \xi(T_u) = x(u) + iy(u)$ and

$$\psi(u) = \frac{\hat{y}}{y(u)}|g'_{T_u(\hat{z})}(\hat{z})|.$$

Notice that $y(u) = e^u$.

Theorem 5.3.2. *Assume $\hat{y} \neq 1$ and put $\nu = -sign(\log \hat{y})$. Let $b \in \mathbb{R}$ and define a, λ by*

$$a = 2b + \nu\kappa b(1-b)/2,\ \lambda = 4b + \nu\kappa b(1-2b)/2.$$

Then

$$F(\hat{z}) = \hat{y}^a E[(1 + x(0)^2)^b |g'_{T_0(\hat{z})}(\hat{z})|^a] = (1 + (\frac{\hat{x}}{\hat{y}})^2)\hat{y}^\lambda.$$

Proof : We consider the function

$$\overline{F}(\hat{z}) = \hat{y}^a E[|g'_{T_0(\hat{z})}(\hat{z})|^a] = E[\psi(0)^a].$$

the strategy consists in finding a PDE satisfied by $\overline{F}$ and to solve it. We achieve the proof in the case $\hat{y} > 1$: we run u between 0 and $\hat{u} = \log(\hat{y})$ and we define $\mathcal{F}_u =< \xi(v)\,;\, v \leq T_u >$.Then an immediate application of the chain rule and Markov property shows that

$$E[\psi(0)^a|\mathcal{F}_u] = \psi(u)^a\overline{F}(z(u)),$$

i.e. the right-hand side is a martingale. Using Itô formula (taking $x, y, \log\psi$ as variables) we find that

$$\frac{4ay^2}{(x^2+y^2)^2}\overline{F} + \frac{2x}{x^2+y^2}\frac{\partial \overline{F}}{\partial x} - \frac{2y}{x^2+y^2}\frac{\partial \overline{F}}{\partial y} + \frac{\kappa}{2}\frac{\partial^2 \overline{F}}{\partial x^2} = 0$$

and it is not difficult to check that

$$\hat{F}(x+iy) = (1+(\frac{x}{y})^2)^b y^\lambda$$

is a solution of this equation. But this solution does not satisfy the initial condition $\hat{F} = 1$ for $y = 1$. It follows that we get a formula for F rather than for $\overline{F}$. Assuming then that F is C^2 the above reasonning remains true for F, and the theorem is thus proven after admitting that F is smooth.
We now put to use this theorem to give bounds for $\hat{f}_t'$:

Theorem 5.3.3. *Let $b \in [0, 1+\frac{4}{\kappa}]$: there is a constant $c(\kappa, b)$ such that*

$$P(|f_t'(x \hat{+} iy)| \geq \frac{\delta}{y}) \leq c(\kappa, b)(1+(\frac{x}{y})^2)^b(\frac{y}{\delta})^\lambda \theta(\delta, a-\lambda)$$

with

$$\theta(\delta, s) = \begin{cases} \delta^{-s} & \text{if } s > 0 \\ 1+|\log\delta| & \text{if } s = 0 \\ 1 & \text{if } s < 0 \end{cases}$$

Proof : We know that $\hat{f}_t'$ has the same distribution as g_{-t}'. Now if $u_1 = \log\Im(g_{-t}(x+iy))$ we can write, since

$$|\frac{\partial}{\partial u}(\log|g_t'|)| = \frac{\Re((g_t-\xi(t))^2}{|g_t-\xi(t)|^2} \leq 1$$

,

$$|fracg_{-t}'(z)g_{T_u}'(z)| \leq e^{|u-u_1}|.$$

It follows that

$$P(|g_{-t}'(z)| \geq \delta/y) \leq C\sum_{j=[\log y]}^{0} P(|g_{T_j}'(z)| \geq \delta/y).$$

Now by scaling

$$E[y^a e^{-ja}|g_{T_j}'(z)|^a] \leq F(e^{-jz)}$$

and the result follows by application of Tchebychev inequality.
We are now about to conclude the proof of the theorem : define $H(y,t) = \hat{f}_t(iy)\,, y > 0\,, t \geq 0$.

Theorem 5.3.4. *If $\kappa \neq 8$ then H extends continuously to $[0, +\infty[\times[0, +\infty[$.*

This theorem will follow from the next proposition, which needs some notation. Let $j, k \in \mathbb{N}, k < 2^{2j}$ and $R_{j,k} = [2^{-j-1}, 2^{-j}] \times [k2^{-2j}, (k+1)2^{-2j}]$. We also define

$$d(j,k) = \operatorname{diam}(H(R_{j,k})).$$

Proposition 5.3.1. *Let $b = \frac{\kappa+8}{4\kappa}$ and a, λ as before with $\nu = 1$. If $\kappa \neq 8$, $\lambda > 2$ and we choose $0 < \sigma < \frac{\lambda-2}{\max(a,\lambda)}$. Then*

$$\sum_{j\geq 0, 0\leq k\leq 2^{2j}-1} P(d(j,k) \geq 2^{-j\sigma}) < \infty.$$

The proof of this proposition is rather technical and we will not give it here. We prefer to give the main ideas behind it. Assuming that $d(j,k) \sim 2^{-j}|\hat{f}_t'(i2^{-j})|$ (the main technical part consists in showing that this is indeed the case), the last theorem shows that

$$P(d(j,k) \geq 2^{-j\sigma}) \leq \begin{cases} 2^{j\sigma a - j\lambda} & , a > \lambda \\ 2^{j\sigma\lambda - j\lambda} & , a < \lambda \end{cases}$$

and we get the result if we can show that $\lambda > 2$. But λ is precisely maximal for $b = (\kappa + 8)/4\kappa$: its value is then $(8+\kappa)^2/16\kappa$ which is minimal for $\kappa = 8$, where it is equal to 2. This explains in particular why the method does not allow to reach $\kappa = 8$.

Definition 5.3.1. *A chordal SLE path is a random curve γ that generates chordal SLE_κ.*

In particular, if γ is a SLE path then

$$g_t(\gamma(t)) = \sqrt{\kappa} B_t$$

This value of g_t has of course to be interpreted as a proper limit.

5.4 Phases for SLE.

An important remark is that (5.2.1) remains valid if $z \in \mathbb{R}$ and that the solution is then real (and may stop to exists after time T_z as for all starting points). The importance of this remark will be clear after we reinterpretate the following calculation in the case $z \in \mathbb{R}$. Put

$$\hat{g}_t(z) = \frac{g_t(z) - \sqrt{\kappa} B_t}{\sqrt{\kappa}} :$$

then $\hat{g}_t(\gamma(t)) = 0$ and $\hat{g}_t(z)$ satisfies the following SDE :

$$dX_t = \frac{2/\kappa}{X_t}dt - dB_t$$

which is Bessel equation. This remark is the key to the following theorem which gives a description of the phase transitions of the family of SLE's :

Theorem 5.4.1. *According to the different values of κ we have the following phases :*

1. *If $0 \leq \kappa \leq 4$ γ is a single curve such that $\gamma(0, +\infty) \subset \mathbb{H}$ and $\lim_{t\to\infty} \gamma(t) = \infty$.*
2. *If $4 < \kappa < 8$ then with probability $1, \bigcup_{t>0} \overline{K}_t = \overline{\mathbb{H}}$ but $\gamma([0, \infty[) \cap \mathbb{H} \neq \mathbb{H}$. Also,$\lim_{t\to\infty} \gamma(t) = \infty$.*
3. *If $\kappa \geq 8$ then γ is a space filling curve, i.e. $\gamma[0, +\infty) = \overline{\mathbb{H}}$.*

Proof : It starts with the

Lemma 5.4.1. *For $x > 0$ we recall that T_x stands for the life-time of $g_t(x)$ which is the same as the first time $\hat{g}_t(z) = 0$. We then have*

1. *If $\kappa \leq 4$ then a.s.$T_x = \infty, \forall x > 0$.*
2. *If $\kappa > 4$ then a.s. $T_x < \infty, \forall x > 0$.*
3. *If $\kappa \geq 8$ then a.s. $T_x < T_y, \forall 0 < x < y$.*
4. *If $4 < \kappa < 8$ then a.s. $P(\{T_x = T_y\}) > 0, \forall 0 < x < y$.*

Proof : It is just a rephrasement of theorem(5.1.1) with $a = 2/\kappa$.
We now come back to the proof of the theorem. We will need the following notation : if $s \geq 0, \gamma^s(t) = g_s(\gamma(t + s)) - \sqrt{\kappa}B_s$ which has the same distribution as γ. To prove 1) we notice that if $\gamma(t) \in (0, +\infty)$ then $T_{\gamma(t)} < \infty$. Also if $\exists t_1 < t_2$ with $\gamma(t_1) = \gamma(t_2)$ then for every $q \in [t_1, t_2[, \gamma^q(0, \infty) \cap \mathbb{R} \neq \emptyset$,which contradicts the first part of the proof.
Let us prove now that $\underline{\lim}_{t\to\infty}|\gamma(t)| = +\infty$ if $\kappa \leq 4$. Let $\delta \in (0, 1/4), x > 1$ and let $t_\delta = \inf\{t > 0; d(\gamma(t), [1, x]) \leq \delta\}$. Now, obviously,

$$g_{t_\delta}(1/2) - \sqrt{\kappa}B_{t_\delta} = \lim_{y\to\infty} \omega(iy, \mathbb{H}; [\sqrt{\kappa}B_{t_\delta}, g_{t_\delta}(1/2)])$$

$$= \lim_{y\to\infty} \omega(iy, \mathbb{H}_{t_\delta}; \text{the part of } \partial\mathbb{H}_{t_\delta} \text{ between } 1/2 \text{ and } g_{t_\delta}(t_\delta)) \leq C\delta.$$

Now assume first that $\kappa < 4$. Then we know that

$$\lim_{t\to\infty}(g_t(1/2) - \sqrt{\kappa}B_t) = \infty,$$

from which it follows first that $d(\gamma([0, \infty[), [1, x]) > 0$ and then, by scaling, that

$$\forall 0 < x_1 < x_2, d(\gamma([0, \infty[), [x_1, x_2]) > 0.$$

To finish the proof we now consider τ, the hitting time of the unit circle for γ (by scaling if necessary, we may assume it is finite). For all $\varepsilon > 0$ there exists $0 < x_1 < x_2$ such that with probability $\geq 1-\varepsilon$ the two images of 0 under g_τ are in $[\sqrt{\kappa}B_\tau - x_2, \sqrt{\kappa}B_\tau - x_1] \cup [\sqrt{\kappa}B_\tau + x_1, \sqrt{\kappa}B_\tau + x_2]$. It follows from what we have just seen and the strong Markov property that with probability at least $1-\varepsilon$,

$$d(g_\tau(\gamma([\tau, +\infty[) - \sqrt{\kappa}B_\tau, [-x_2, -x_1] \cup [x_1, x_2]) > 0,$$

and finally that $d(0, \gamma([\tau, +\infty[) > 0$. By scaling, the property follows.

Case $\kappa = 4$: Assume $0 < y < x$ and consider the domain D_t whose complement is the union of the half-line $]-\infty, y]$, the curve $\gamma([0,t])$ and its relexion across the real axis. The map g_t extends by Schwarz reflection to a conformal mapping from D_t onto $\mathbb{C}\backslash]-\infty, g_t(y)]$. By Koebe theorem,

$$d(x, \partial D_t) \geq \frac{g_t(x) - g_t(y)}{4g_t'(x)},$$

and it suffices to prove that

$$\sup_{t\geq 0} \frac{g_t'(x)}{g_t(x) - g_t(y)} < +\infty.$$

To this end we denote $\xi(t) = \sqrt{\kappa}B_t$, $Y_{x(y)}(t) = g_t(x(y)) - \xi(t)$ and

$$Q(t) = \log(g_t'(x)) - \log(g_t(x) - g_t(y)).$$

Then

$$\frac{\partial}{\partial t}Q(t) = -2Y_x(t)^2 + 2Y_x(t)^{-1}Y_y(t)^{-1}$$

which shows in particular that $Q(t)$ is nondecreasing. We define now the function

$$G(s) = \log(s)\log(1+s) - \frac{1}{2}\log^2(1+s) + \int_{-s}^{0} \frac{\log(1-u)}{u} :$$

this function is a solution of the differential equation

$$s(1+s)^2 G''(s) + s(1+s)G'(s) = 1$$

and also happens to be bounded on $]0, +\infty[$. By Itô formula

$$Q(t) - G(\frac{g_t(x) - g_t(y)}{g_t(y) - \xi(t)})$$

is a local martingale. There thus exists a sequence (t_n) of stopping times increasing to $+\infty$ such that

$$E[Q(t_n)] = E[G(t_n)] + Q(0) - G_0.$$

Since G is bounded, this implies that

$$\overline{\lim}_{n\to+\infty} E[Q(t_n] < +\infty.$$

Using then the fact that Q is nondecreasing and the monotone convergence theorem we get successively that $E[\sup_t Q(t)] < +\infty$ and $\sup_t Q(t) < +\infty$ a.s. Case $4 < \kappa < 8$: We will say that a point $z \in \mathbb{H}$ is swallowed if $T_z < \infty$ but $z \notin \bigcup_{t<T_z} \overline{K}_t$. Swallowed points form an open set and lifetime is constant in each connected component. By lemma(5.4.1) there is a positive probability that for some $x > 1, T_x = T_1$. In fact, by an easy scaling argument, this probability is equal to 1 and $\gamma(T_1)$ is the largest real x with $T_x = T_1$. Let $\varepsilon = d(1, \gamma([0, T_1])$.Then all points in $\mathbb{H} \cap B(1, \varepsilon)$ are swallowed :this shows that the curve γ does not fill the half-plane. Also let T be the first time that both $1, -1$ are swallowed ; then there exists a disk centered at 0 whose intersection with the half-plane is included in K_T. Thus for every u there exists $\varepsilon, t = t(\varepsilon, u)$ such that with probability $\geq 1 - u$, $B(0, \varepsilon) \cap \mathbb{H} \subset K_t$. y scaling this must hold for all ε. This implies that $d(0, \mathbb{H}\backslash K_t) \to \infty$ and in particular that $\gamma(t) \to \infty$ as $t \to \infty$.
Case $\kappa \geq 8$. Notice first that lemma(5.4.1) shows that every real point belongs to the curve γ. Let us now prove the same for every point of the half-plane. First of all there cannot be any swallowed point in this case since there cannot be any real swallowed point from the fact that $T_x < T_y$ if $x < y$. It follows that $K_t = \gamma([0, t])$. The result will then follow if we can prove that the random variable

$$\Delta(x) = d(x + i, \gamma([0, +\infty[)$$

is identically equal to 0. To this end we change a little the notations. Writing $a = 2/\kappa$, $h_t(z) = g_t(\sqrt{\kappa}z)/\sqrt{\kappa}$, then h_t satisfies the Löwner equation

$$\dot{h}_t(z) = \frac{a}{h_t(z) + B_t}$$

and $Z_t = h_t + B_t$ satisfies the Bessel type equation

$$dZ_t = \frac{a}{Z_t} dt + dB_t.$$

We write $Z_t = X_t + iY_t$ and we consider the time-change defined by

$$t = \int_0^{\sigma(t)} \frac{ds}{X_s^2 + Y_2^s},$$

which really means that time becomes a function of Y. If then A_t is any process linked with the problem we put $\tilde{A}_t = A_{\sigma(t)}$. Suppose now that the curve does not fill the half-plane. Then by scaling we may assume that there exists $x \in \mathbb{R}, \Delta(x) \neq 0$ and $T(x + i) = +\infty$ by the above discussion. By Koebe theorem, $\Delta(x)$ is comparable to $e^{-D(x)}$ where

$$D(x) = \lim_{t\to\infty} \ln \frac{|h_t'(x + i)|}{\Im(h_t(x + i))}.$$

Put

$$D_t(z) = \ln \frac{h'_t(z)}{\Im(h_t(z))}.$$

An easy computation shows that

$$\partial_t(\ln |h'_t)| = a\frac{Y_t^2 - X_t^2}{(X_t^2 + Y_t^2)^2}$$

while

$$\partial_t(\ln \Im(h_t)) = -a\frac{1}{X_t^2 + Y_t^2}.$$

Finally

$$\partial_t(D_t) = \frac{2aY_t^2}{(X_t^2 + Y_t^2)^2}$$

and thus

$$D(x) = 2a\int_0^{+\infty} \frac{Y_t^2}{(X_t^2 + Y_t^2)^2}dt.$$

Let $D_t(x)$ be the integral from 0 to t and putting $K_t = (X_t/Y_t)$, $C_t = \ln K_t$, we see with the help of Itô's formula, that

$$d\tilde{C}_t = [2a - \frac{1}{2} - \frac{1}{2}e^{-2\tilde{C}_t}]dt + \sqrt{1 + e^{-2\tilde{C}_t}}d\tilde{B}_t$$

and

$$\dot{\tilde{D}}_t = \frac{2a}{1 + e^{-2\tilde{C}_t}} \Rightarrow D(x) = \int_0^{\infty} \frac{2a}{1 + e^{-2\tilde{C}_t}}dt.$$

The fact that $\kappa \geq 8$ corresponds to the fact that $a \leq 1/4$, in which case the drift term is negative. This last fact implies that whatever large is $T > 0$ there exists $t > T$ such that $\tilde{C}_s \leq 0, s \in [t, t+1]$. But this implies that $D(x) = +\infty$ and the proof is complete.

5.5 Transience

Theorem 5.5.1. *Let $\kappa \neq 8$ and γ be the generating curve. Then*

$$\lim_{t \to +\infty} |\gamma(t)| = +\infty.$$

Proof : We have already seen this property for $\kappa < 8$. We assume $\kappa > 8$: it suffices to show, using some $0, 1$ law that there exists a positive t such that with positive probability $0 \neq \overline{\Omega_t}$. Arguing by contadiction, we assume that for all $t > 0$, a.s. $0 \in \overline{\Omega_t}$. By Markov property and the fact that γ fills $\mathbb{H}$ it then follows that the same is true for all $z \in \mathbb{H}$ and consequently that ∂K_t has positive area. But this contradicts the

Theorem 5.5.2. *For every $\kappa \neq 4$ and every $t > 0$ the mapping f_t is Hölder continuous on $\mathbb{H}$.*

It is well known that this implies that $\partial\Omega_t$ has dimension < 2 and in particular 0-area.
Proof : Put

$$z_{j,n} = (j+i)2^{-n},\ 0 \leq n < +\infty,\ -2^n < j < 2^n,$$

and let us try to estimate $|\hat{f}_t(z_{j,n})|$: we use to this end theorem(5.3.3) with $\delta = 2^{-nh}$ to get

$$P[|\hat{f}_t(z_{j,n})| \geq 2^{n(1-h)} \leq C(\kappa, b)(1+2^{2n})^b 2^{-n(1-h)\lambda}\theta(2^{-nh}, a-\lambda).$$

Hence

$$\sum_{n,j} P[|\hat{f}_t(z_{j,n})| \geq 2^{n(1-h)} < \infty$$

provided that

$$1 + 2b - (1-h)\lambda < 0 \,\text{and}\, a - \lambda \leq 0$$

or that

$$1 + 2b - \lambda + ah < 0 \,\text{and}\, a - \lambda \geq 0.$$

If $0 < \kappa \leq 12$ and $b = 1/4 + 1/\kappa$, $h < (\kappa-4)^2/((\kappa+4)(\kappa+12))$ the first condition is satisfied. For $\kappa > 12$, $b = 4/\kappa$, $h < 1/2 - 4/\kappa$ the second condition is satisfied. An application of Borel-Cantelli lemma and of Koebe theorem then shows the theorem.

5.6 dimension of SLE paths

In this section we will try to convince the reader that the box-dimension of $SLE_{kappa}-$paths is $1 + \frac{\kappa}{8}$ if $\kappa < 8$. This will follow from the following theorem, in which the notations are those of the last paragraph :

Theorem 5.6.1. $P(\Delta(x) \leq \varepsilon) \sim \varepsilon^{1-\frac{\kappa}{8}}$ *if* $\kappa < 8$.

Proof : We use the notations of section 5.4. To estimate this probability one computes explicitely the characteristic function

$$E\left[e^{ibD(x)}\right].$$

To do so we put $K_t = X_t/Y_t$: we performed a change of variable in the last section, leading to $\tilde{K}_t$. One here perform a second one, namely $\hat{\sigma}'(t) = (\tilde{K}^2_{\hat{\sigma}(t)} + 1)^{-1}$, leading to

$$d\hat{K}_t = \frac{2a\hat{K}_t}{1+\hat{K}_t^2}dt + d\hat{B}_t, dD_t(x) = \frac{2a\hat{K}_t}{(1+\hat{K}_t^2)^2}.$$

We now seek for a function ψ such that $\psi(\hat{K}_t)e^{ibD_t(x)}$ is a local martingale. An application of Itô's formula shows that the function ψ must be solution of the ODE

$$\frac{1}{2}y'' + \frac{2ax}{1+x^2}y' + \frac{ib}{(1+x^2)^2}y = 0.$$

To solve this equation we change the variable and look for a solution of the form

$$H(\frac{x^2}{x^2+1})$$

and it happens that H must be a solution of the hypergeometric equation

$$u(1-u)H''(u) + [\frac{1}{2} + 2(a-1)u]H'(u) + \frac{1}{2}abiH(u) = 0.$$

There exists a solution to this equation which is bounded and such that $H(1) = 1$:

$$H(u) = cF(\alpha_+, \alpha_-, \frac{1}{2}, u)$$

where

$$F(\alpha, \beta, \gamma, z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} z^k$$

$((\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1))$ and

$$c = \frac{\Gamma(a + \sqrt{(a-1/2)^2 - iab})\Gamma(a - \sqrt{(a-1/2)^2 - iab})}{\Gamma(1/2)\Gamma(2a-1/2)}.$$

The idea is now tu use the optional stopping theorem for the martingale

$$M_t = \psi(Y_t e^{ibD_t})$$

in the form

$$E[M_0] = E[M_\infty]$$

. Using the fact that $\hat{K}_t \to +\infty$ a.s. and $\lim_{x\to\infty} \psi(x) = 1$ we get

$$E[e^{ibD(x)}] = H(\frac{x^2}{x^2+1}).$$

Using properties of the function Γ we get that

$$E[e^{ibD(0)}] = \frac{c}{1 - \frac{\kappa}{8} - ib} + v(b)$$

where v is analytic in $\{|z| < 1 - \kappa/8 + \varepsilon\}$ for some $\varepsilon > 0$. The proof of the theorem will be achieved (for $x = 0$, the general case being similar) by application of the following analysis lemma, whose proof is left to the reader :

Lemma 5.6.1. *Suppose X is a random variable with characteristic function Φ satisfying for some $u, \lambda, \varepsilon > 0$*

$$\Phi(t) = \frac{u\lambda}{\lambda - it} + v(t),$$

where v is analytic on $\{|z| < \lambda + \varepsilon\}$, then

$$P[X \geq x] = ue^{-\lambda x} + o(e^{-\lambda x}).$$

5.7 Locality for SLE_6

In this section we consider $K_t, t \geq 0$ a chordal SLE_κ.For convenience we will write $W_t = \sqrt{\kappa} B_t$. We also consider a hull A which is at positive distance from 0. Let Φ be the normalized conformal mapping from $\mathbb{H} \backslash A$ onto $\mathbb{H}$. Let T be the first time that K_t intersects A. For $t \leq T$ we can define $\tilde{K}_t = \Phi(K_t)$. The goal of this section is to compare the growth of K_t and $\tilde{K}_t$.
Let Φ_t be the normalized Riemmann mapping from $\mathbb{H} \backslash g_t(A)$ onto $\mathbb{H}$ where g_t is the Löwner process describing K_t (notice that $\Phi = \Phi_0$). Then, if $\tilde{g}_t$ is the Löwner process describing $\tilde{K}_t$, we have

$$\Phi_t \circ g_t = \tilde{g}_t \circ \Phi_0.$$

Write $\tilde{W}_t = \Phi_t(W_t)$ so that the differential equation satisfied by $\tilde{g}_t$ reads

$$\partial_t \tilde{g}_t(z) = \frac{2\partial_t(\text{hcap}(\tilde{K}_t))}{\tilde{g}_t(z) - \tilde{W}_t}.$$

It remains to understand the evolution of $\text{hcap}(\tilde{K}_t))$ and $\tilde{W}_t$.
For the first quantity we write, for $0 < s < t$, $g_t = g_{s,t} \circ g_s$ and parallely $\tilde{g}_t = \tilde{g}_{s,t} \circ \tilde{g}_s$. Then we can write $\text{hcap}(\tilde{K}_t) = \text{hcap}(\tilde{K}_s) + \text{hcap}(\tilde{K}_{s,t})$ where $\tilde{K}_{s,t} = \tilde{g}_s(\tilde{K}_t)$ and

$$\lim_{t \to s} \frac{\text{hcap}(\tilde{K}_{s,t})}{t - s} = \Phi_s'^2(W_s)$$

because of the scaling property of hcap.
In order to evaluate the second quantity we start with the identity

$$\Phi_t = \tilde{g}_t \circ \Phi \circ g_t^{-1}$$

that we differentiate wrt t. Using the inverse Löwner equation

$$\partial_t(g_t^{-1}(z)) = -2 \frac{(g_t^{-1})'(z)}{z - W_t}$$

from which it is easy to deduce that

$$\partial_t \Phi_t(z) = \frac{2\Phi_t'(W_t)^2}{\Phi_t(z) - \tilde{W}_t} - \frac{2\Phi_t'(z)}{z - W_t}.$$

By Schwarz reflection the time derivative of $\Phi_t(z)$ exists for $z = W_t$ and we must have

$$(\partial_t \Phi_t)(W_t) = \lim_{z \to W_t} [\frac{2\Phi_t'(W_t)^2}{\Phi_t(z) - \tilde{W}_t} - \frac{2\Phi_t'(z)}{z - W_t}] = -3\Phi_t''(W_t).$$

We finally make use of Itô's formula which gives :

$$d\tilde{W}_t = (\partial_t \Phi_t)(W_t)dt + \Phi_t'(W_t)dW_t + \frac{\kappa}{2}\Phi_t''(W_t)dt$$

hence,

$$d\tilde{W}_t = \Phi_t'(W_t)dW_t + [\frac{\kappa}{2} - 3]\Phi_t''(W_t)dt.$$

We can now state the main result of this section :

Theorem 5.7.1. *If $\kappa = 6$ then, modulo time-change, the process $\tilde{K}_t - \Phi(0), t < T$ has the same law as K_t.*

Proof : The time-change is of course $\mathrm{hcap}(K_t) = \int_0^t \Phi_s'(W_s)^2 ds =< \tilde{W} >_t$. Hence if we define $\tilde{W}_t = \hat{W}_{\mathrm{hcap}(K_t)}$ then $\hat{W} - \hat{W}_0$ and W have the same law. Moreover, if we define $\hat{g}$ by $\tilde{g}_t = \hat{g}_{\mathrm{hcap}(K_t)}$ we have

$$\partial_t \hat{g}_t(z) = \frac{2}{\hat{g}_t(z) - \hat{W}_t}.$$

5.8 Restriction Property for $SLE_{8/3}$

In this section we keep the same notations as in the preceeding one ; we would like to understand the evolution of $\Phi_t'(W_t)$. To this end we differentiate the equation (5.7) :

$$\partial_t \Phi_t'(z) = -\frac{2\Phi_t'(W_t)^2 \Phi_t'(z)}{(\Phi_t(z) - \tilde{W}_t)^2} + \frac{2\Phi_t'(z)}{(z - W_t)^2} - \frac{2\Phi_t''(z)}{z - W_t}.$$

Taking the limit as $z \to W_t$ we obtain :

$$\partial_t \Phi_t'(W_t) = \frac{\Phi_t''(W_t)^2}{2\Phi_t'(W_t)} - \frac{4}{3}\Phi_t'''(W_t).$$

If we then apply Itô's formula, we get

$$d[\Phi_t'(W_t)] = \Phi_t''(W_t)dW_t + \left[\frac{\Phi_t''(W_t)^2}{2\Phi_t'(W_t)} + (\kappa/2 - 4/3)\Phi_t'''(W_t)\right] dt.$$

From now on in this section we specialize $\kappa = 8/3$. Put $X_t = \Phi_t'(W_t)$. We look for an index α such that X_t^α is a local martingale (in fact a bounded martingale in this case since $X_t \leq 1$). Applying Itô's formula we see that $\alpha = 5/8$ does the job and that

$$d\left[\Phi_t'(W_t)^{5/8}\right] = \frac{5\Phi_t''(W_t)}{8\Phi_t'(W_t)^{3/8}} dW_t.$$

We can now state

Proposition 5.8.1. *For chordal $SLE_{8/3}$ and any hull A not containing* 0*,*

$$P(\forall t \geq 0, K_t \cap A = \emptyset) = \Phi_A'(0)^{5/8}.$$

Proof : Let us denote by M_t the local martingale $\Phi_t'(W_t)^{5/8}, t \leq T$. First of all notice that this is actually a martingale bounded by 1. Indeed, if we denote by u, v the real and imaginary parts of Φ_t, then by parity $\partial u/\partial y$ is equal to 0 on the real line while $\partial v/\partial y \in [0,1]$ also on the real line since one easily sees by maximum principle that $v(z) \leq y$ on $\mathcal{H}$. It is not difficult to see that if $T = \infty$ then, if τ_R stands for the hitting time of the circle centered at 0 with radius R,

$$\lim_{R\to\infty} \Phi_{\tau_R}'(W_{\tau_R}) = 1.$$

On the other hand, if $T < +\infty$ then $lim_{t\to T}\Phi_t'(W_t) = 0$. It follows that

$$P(T = \infty) = E[M_T] = E[M_0] = \Phi_A'(0)^{5/8}.$$

We can now state the theorem about the restriction property :

Theorem 5.8.1. *Suppose that A_0 is a hull; then the conditionnal law of $K_\infty = \cup_{t>0} K_t$ given $K_\infty \cap A_0 = \emptyset$ is identical to the law of $\Psi_{A_0}(K_\infty)$, where Ψ_A is a Φ_A translated so that $\Psi_A(0) = 0$.*

Proof : The law of K_∞ is characterized by the knowledge of $P(K_\infty \cap A = \emptyset)$ for all hulls A not containing 0. Let A be such a hull :

$$P(\Psi_{A_0}(K_\infty) \cap A = \emptyset | K_\infty \cap A_0 = \emptyset)$$

$$= \frac{P(K_\infty \cap \mathbb{H}\backslash \Psi_{A_0}^{-1} \circ \Psi_A^{-1}(\mathbb{H}) = \emptyset)}{P(K_\infty \cap A_0 = \emptyset)}$$

$$= \left(\frac{\Psi_{A_0}'(0)\Psi_A'(0)}{\Psi_{A_0}'(0)}\right)^{5/8} = P(K_\infty \cap A = \emptyset).$$

5.9 The Mandelbrot conjecture : outline of a proof

The Brownian frontier is the boundary of the unbounded component of $\mathbb{C}\backslash B[0,1]$, where B_t is a planar Brownian motion. Mandelbrot conjectured that this set has Hausdorff dimension 4/3. This conjectured has been proved by Lawler, Schramm, Werner using SLE. More precisely they proved it by connecting it with $SLE_{8/3}$ whose dimension is precisely 4/3 by what we have seen preceedingly.
The proof undergoes the notion of Brownian excursion from 0 to ∞ in $\mathbb{H}$. This process can be seen as a Brownian motion conditionned to stay in $\mathbb{H}$. It is defined as $W = X + iY$ where X, Y are independent real processes, X being a standard Brownian motion, and Y being a $3-$dimensional Bessel process. If T_r denotes the first passage of W at height r then the law of $W([T_r, T_R]) - W(T_r)$ is the law of a Brownian motion started at ir stopped when it hits $\mathbb{R} + iR$ and conditionned to stay in the upper-half-plane up to this time. Note that this event has probability r/R.

Theorem 5.9.1. *Suppose A is a hull at positive distance from 0 and let W be a Brownian excursion from 0 to ∞ in $\mathbb{H}$. Then*

$$P[W([0,+\infty[) \cap A = \emptyset] = \Phi'_A(0).$$

Proof : Let W, Z be respectively an Brownian excursion and a planar Brownian motion starting at $z \in \Phi^{-1}(\mathbb{H})$. Since $\Im(\Phi(z)) - \Im(z) \to 0$ as $\Im(z) \to \infty$ we may write

$$P[\Phi(Z)([0, T_R(Z)]) \subset \mathbb{H}) \sim P[\Phi(Z)([0, T_R(\Phi(Z))]) \subset \mathbb{H}).$$

But by conformal invariance of Brownian motion the right-hand side is equal to $\Im(\Phi(z))/R$ so that

$$P[W([0, T_R(W)]) \subset \Phi^{-1}(\mathbb{H})) =$$
$$\frac{P[Z([0, T_R(Z)]) \subset \Phi^{-1}(\mathbb{H})]}{P[Z([0, T_R(Z)]) \subset \mathbb{H})]} = \frac{\Im(\Phi(z))}{\Im(z)} + o(1)$$

as $R \to \infty$. Letting $z \to 0$ we then get the result, since $\Phi(z) = z\Phi'(0) + o(|z|)$ at 0.

Theorem 5.9.2. *Let F_8 denotes the "'filling"' of 8 independent chordal $SLE_{8/3}$ and F_5 the filling of 5 independent Brownian excursions. The F_8 and F_5 have the same law.*

Notice that this theorem proves Mandelbrot conjecture.
Proof : The law of both processes is characterized by the probabilities $P[F \cap A = \emptyset]$ but in both cases they are equal to $\Phi'_A(0)^5$.

Chapitre 6

Hele-Shaw Flows and Aggregation Processes

6.1 Hele-Shaw Flows

We study a flow of an incompressible fluid between two parallel plates which are fixed at a small distance h. Neglecting gravity, the Navier-Stokes equations read

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = \frac{1}{\rho}(-\nabla p + \mu \Delta V),$$

$$\nabla \cdot V = 0.$$

In these equations, V is the time-dependent vector-field of velocities, ρ is the density and μ the viscosity coefficient.
We consider very slow flows. We may thus assume that $\frac{\partial V}{\partial t} = 0$, $V_3 = 0$. We then get

$$(V_1 \frac{\partial}{\partial x_1} + V_2 \frac{\partial}{\partial x_2})V_j = \frac{1}{\rho}(\frac{\partial p}{\partial x_j} + \mu \Delta V_j),\ j = 1, 2,$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x_3}$$

with the boundary conditions

$$V_1, V_2 = 0 \,\text{on}\, x_3 = 0, h.$$

With h very small and the flow sufficiently slow one can furher neglect $\frac{\partial V}{\partial x_{1}, x_{2}}$ in front of $\frac{\partial V}{\partial x_3}$.
The system becomes :

$$\frac{\partial p}{\partial x_j} = \mu \frac{\partial^2 V_j}{\partial^2 x_3},\ j = 1, 2,$$

$$0 = \frac{\partial p}{\partial x_3}.$$

This last equations says that p does not depend on x_3. With the boundary conditions, it implies that

$$V_j = \frac{1}{2}\frac{\partial p}{\partial x_j}(\frac{x_3^2 - hx_3}{\mu}) \text{ , } j = 1, 2.$$

We consider now the integral means

$$\overline{V_j} = \frac{1}{h}\int_0^h V_j dx_3.$$

We obtain the Hele-Shaw (HS) equation :

$$\overline{V} = -\frac{h^2}{12\mu}\nabla p.$$

This is a two dimensional potential flow for which the potential is proportional to the pressure. The incompressibility assumption implies that

$$\nabla \cdot \overline{V} = 0$$

so that pressure is an harmonic function outside the source/sink.
This source or sink is at 0 and is assumed to be of constant strength. The rate of mass-change is given by

$$\int_{\partial D(0,\varepsilon)} \rho V \cdot n ds = Cte.$$

Using Green-Riemann and Hele-Shaw equation we have

$$\int\int_{D(0,\varepsilon)} \Delta p dx_1 dx_2 = Cte$$

which implies that $\Delta p = Q\delta_{\Omega - t}(., 0)$, where Ω_t is the domain occupied by the viscous fluid at time t.
On the other hand the pressure on the boundary is equal to the exterior air pressure which we may assume to be constant plus surface tension that we will also assume to be constant. Since the pressure only appears through its gradient in the equation we will assume that the constant value of the pressure on the boundary is 0 so that the pressure function is Q times the Green's function at 0. The case $Q < 0$ corresponds to injection while $Q > 0$ is the sucction case.

6.1.1 Polubarinova-Galin Equation

These two russian mathematicians were the first to write down HS-equation in terms of the Riemann mappings $f(.,t)$ from the unit disk onto Ω_t fixing 0. We will consider the physical plane to be the z-one while the unit disk will lie in the ζ-plane. The tangent to $\partial\Omega_t$ at point z is given by $i\zeta f'(\zeta)/|f'(\zeta)|$ so that the exterior normal is given by $n = \zeta f'(\zeta)/|f'(\zeta)|$. The pressure is given by $p(z) = -\log|f_t^{-1}(z)| = -\log|\zeta|$ so that

$$\nabla p = -\frac{Q}{\overline{\zeta f'(\zeta,t)}}, \; \frac{\partial p}{\partial n} = -\frac{Q}{|f'|}$$

while the normal velocity is given by

$$v_n = \Re(\dot{f}\frac{\overline{\zeta f'}}{|f'|}.$$

Since Hele-Shaw equation implies that $\frac{\partial p}{\partial n} = v_n$ we can derive Polubarinova-Galin equation :

$$\Re(\dot{f}\overline{\zeta f'}) = -Q.$$

This equation is equivalent to a Löwner type equation. Indeed, for ζ on the circle,

$$\frac{\dot{f}}{\zeta f'} = \frac{\dot{f}\overline{\zeta f'}}{|f'|^2} = \frac{Q}{|f'|^2}.$$

Polubarinova-Galin equation is thus equivalent to the following (implicit) Löwner equation

$$\dot{f}(\zeta,t) = -\zeta f'(\zeta,t)\int_{\partial\mathbb{D}} \frac{Q}{|f'|^2(u)}\frac{u+\zeta}{u-\zeta}|du|.$$